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A Common Unique Random Fixed Point Theorem With Rational Inequality In Hilbert Space

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ABSTRACT

We prove a new common fixed point theorem with rational inequality for two random operators defined on a non- empty closed subset of a separable Hilbert Space. Our results generalize and extend the result of Choudhury [2].

Keywords

Common fixed point, random operators, separable Hilbert Space.

AMS Subject Classification (2000): *47H10, 54H25.*

1. INTRODUCTION

The random fixed point theory is started by Prague school of Probabilists in 1950 [6, 9]. In recent years this theory has attracted much attention of many mathematicians some of them are [1, 2, 3, 4, and 8].

In this paper we find a new common fixed point theorem with rational inequality for two random operators defined on separable Hilbert spaces. For this we construct a sequence of measurable function of random fixed point to the two random operators.

2. PRELIMINARY NOTES

Let H be a Hilbert space and C is a closed subset of H. Let (Ω, Σ) be measurable space.

Definition 2.1

A function $f: \Omega \to C$ is called measurable if $f^{-1}(B \cap C) \in \Sigma$ for each Borel subset B of H.

Definition 2.2

A function $F: \Omega \times C \to C$ is called random operator if $F(., x): \Omega \to C$ is measurable for all $x \in C$.

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Definition 2.3

A measurable function $g: \Omega \to C$ is called a random fixed point to the random operator $F: \Omega \times C \to C$ if $F(t, g(t)) = g(t)$ for all $t \in \Omega$.

Definition 2.4

A random operator $F: \Omega \times C \to C$ is called continuous if for fixed $t \in \Omega$, $F(t, .): C \to C$ is continuous.

Condition

Let C be the any non-empty closed subset of a Hilbert Space H then two mappings $S, T: C \to C$ is said to satisfy Condition if

$$\|Sx - Ty\|^2 \le \alpha\|x - y\|^2 + \beta(\|x - Sx\|^2 + \|y - Ty\|^2) + \gamma(\|x - Ty\|^2 + \|y - Sx\|^2) + \delta \frac{\|y - Sx\|^2}{1 + \|x - Sx\|^2 \|x - Ty\|^2}$$

Where α, β, γ and δ are non-negative real and $0 < \alpha + 2\beta + 4\gamma + 4\delta < 1$.

3. MAIN RESULTS

Theorem 3.1

Let C be a non-empty closed subset of a Separable Hilbert space H. Let S and T be two continuous random operators defined on C such that for $t \in \Omega, S(t, .), T(t, .): C \to C$ satisfy Condition (1.1), then S and T have a unique common fixed point in C.

Proof

We construct a sequence of functions $\{g_n(t)\}$ as $g_0: \Omega \to C$ is arbitrary measurable function for $t \in \Omega$ and $n = 1,2, \dots \dots \dots$

$g_{2n+1}(t) = S(t, g_{2n}(t))$ and $g_{2n+2}(t) = t(t, g_{2n+1}(t))$

$\|g_{2n+1}(t) - g_{2n}(t)\|^2 = \|S(t, g_{2n}(t)) - T(t, g_{2n-1}(t))\|^2$

$$\begin{aligned}
&\le \alpha\|g_{2n}(t) - g_{2n-1}(t)\|^2 + \beta\left(\left\|g_{2n}(t) - S(t, g_{2n}(t))\right\|^2 + \left\|g_{2n-1}(t) - T(t, g_{2n-1}(t))\right\|^2\right) \\
&+ \gamma\,(\|g_{2n}(t)) - T(t, g_{2n-1}(t))\|^2 + \|g_{2n-1}(t)) - S(t, g_{2n}(t))\|^2) \\
&+ \delta \frac{\|g_{2n-1}(t)) - S(t, g(t))\|^2}{1 + \|g_{2n}(t)) - S(tg_{2n+1}(t))\|^2 \|g_{2n}(t)) - T(t, g_{2n-1}(t))\|^2} \\
&= \alpha\|g_{2n}(t) - g_{2n-1}(t)\|^2 + \beta(\|g_{2n}(t) - g_{2n+1}(t)\|^2 + \|g_{2n-1}(t) - g_{2n}(t)\|^2) \\
&+ \gamma(\|g_{2n}(t)) - g_{2n}(t))\|^2 + \|g_{2n-1}(t)) - g_{2n+1}(t))\|^2) \\
&+ \delta \frac{\|g_{2n-1}(t)) - g_{2n+1}(t))\|^2}{1 + \|g_{2n}(t)) - g_{2n+1}(t))\|^2 \|g_{2n}(t)) - g_{2n}(t))\|^2}
\end{aligned}$$

$$= \alpha\|g_{2n}(t) - g_{2n-1}(t)\|^2 + \beta(\|g_{2n}(t) - g_{2n+1}(t)\|^2 + \|g_{2n-1}(t) - g_{2n}(t)\|^2) + \gamma\|g_{2n-1}(t)) - g_{2n+1}(t))\|^2 + \delta\|g_{2n-1}(t)) - g_{2n+1}(t))\|^2$$

$$= (\alpha + \beta)\|g_{2n}(t) - g_{2n-1}(t)\|^2 + \beta\|g_{2n}(t) - g_{2n+1}(t)\|^2 + (\gamma + \delta)\|g_{2n-1}(t)) - g_{2n+1}(t))\|^2$$

$$\begin{aligned}
&\le (\alpha + \beta)\|g_{2n}(t) - g_{2n-1}(t)\|^2 + \beta\|g_{2n}(t) - g_{2n+1}(t)\|^2 + (\gamma + \delta)(2\,\|g_{2n+1}(t)) - g_{2n}(t))\|^2 + 2\|g_{2n}(t) - g_{2n-1}(t)\|^2) \\
&= (\alpha + \beta + 2\gamma + 2\delta)\|g_{2n}(t) - g_{2n-1}(t)\|^2 + (\beta + 2\gamma + 2\delta)\|g_{2n}(t) - g_{2n+1}(t)\|^2
\end{aligned}$$

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$$\Rightarrow \left(1-(\beta+2\gamma+2\delta)\right)\|g_{2n+1}(t)-g_{2n}(t)\|^2$$

$$\leq (\alpha+\beta+2\gamma+2\delta)\|g_{2n}(t)-g_{2n-1}(t)\|^2$$

$$\Rightarrow \|g_{2n+1}(t)-g_{2n}(t)\|^2$$

$$\leq \left(\frac{\alpha+\beta+2\gamma+2\delta}{1-(\beta+2\gamma+2\delta)}\right)\|g_{2n}(t)-g_{2n-1}(t)\|^2$$

For all$t \in \Omega$ and $n = 1,2, \ldots \ldots \ldots$

Similarly,

$\|g_{2n}(t)-g_{2n-1}(t)\|^2$

$$\leq \left(\frac{\alpha+\beta+2\gamma+2\delta}{1-(\beta+2\gamma+2\delta)}\right)\|g_{2n-1}(t)-g_{2n-2}(t)\|^2$$

For all$t \in \Omega$ and $n = 1,2, \ldots \ldots \ldots$

In general for all $t \in \Omega$ and $n = 1,2, \ldots \ldots \ldots$

$$\|g_n(t)-g_{n-1}(t)\|^2 \leq \left(\frac{\alpha+\beta+2\gamma+2\delta}{1-(\beta+2\gamma+2\delta)}\right)\|g_{n-1}(t)-g_{n-2}(t)\|^2$$

And $0 < \left(\frac{\alpha+\beta+2\gamma+2\delta}{1-(\beta+2\gamma+2\delta)}\right) < 1$

It is clear that the sequence $\{g_n(t)\}$ is a Cauchy sequence and hence it is convergent in Hilbert space$H$.

Suppose that $\{g_n(t)\} \rightarrow g(t)$ as $n \rightarrow \infty$ for $t \in \Omega$

Since $g: C \rightarrow C$ is closed.

$$\left\|g(t)-S\left(t,g(t)\right)\right\|^2 = \left\|g(t)-g_{2n}(t)+g_{2n}(t)-S\left(t,g(t)\right)\right\|^2$$

$\leq 2\left\|g(t)-g_{2n}(t)\right\|^2 + 2\|g_{2n}(t)-S\left(t,g(t)\right)\|^2$

$$= 2\left\|g(t)-g_{2n}(t)\right\|^2 + 2\left\|T\left(t,g_{2n-1}(t)\right)-S\left(t,g(t)\right)\right\|^2$$

$$\leq 2\|g(t)-g_{2n}(t)\|^2 + 2\alpha\|g_{2n-1}(t)-g(t)\|^2 + 2\beta\left(\left\|g_{2n-1}(t)-T\left(t,g_{2n-1}(t)\right)k\right\|^2 + \|g(t)-S(t,g(t))\|^2\right)$$
$$+ 2\,\gamma\,(\|g_{2n-1}(t))-S(t,g(t))\|^2 + \|g(t))-T(t,g_{2n-1}(t))\|^2)$$
$$+ \quad \delta\frac{\|g_{2n-1}(t))-S(t,g(t))\|^2}{1+\|g(t))-S(t,g(t))\|^2\|g(t))-T(t,g_{2n-1}(t))\|^2}$$

Letting $n \rightarrow \infty, \{g_{2n}(t)\} \rightarrow g(t)$ we have

$\left\|g(t)-S\left(t,g(t)\right)\right\|^2 \leq (2\beta+2\gamma+\delta)\left\|g(t)-S\left(t,g(t)\right)\right\|^2$, For all $t \in \Omega$

Since $0 < 2\beta+2\gamma+\delta < 1$

Therefore for all $t \in \Omega$, we have $S\left(t,g(t)\right) = g(t)$

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Similarly we can prove that for all $t \in \Omega$. $T(t, g(t)) = g(t)$.

Himmelberg [7] had proved if $G: \Omega \times C \rightarrow C$ is a continuous random operator on closed subset C then for any measurable function $f: \Omega \rightarrow C$ the function $h(t) = G(t, f(t))$, is also measurable function.

Thus $\{g_n(t)\}$ is a sequence of measurable function. And hence g is also a measurable function.

This implies that $g(t)$ is a common random fixed point of S and T.

Uniqueness

Suppose that $h(t): \Omega \rightarrow C$ be another common random fixed point of S and T.

Therefore, for all $t \in \Omega$,

$$S(t, h(t)) = h(t),$$

$$T(t, h(t)) = h(t)$$

Now

$$\|g(t) - h(t)\|^2 = \|S(t, g(t)) - T(t, h(t))\|^2$$

$$\leq \alpha \|g(t) - h(t)\|^2 + \beta \left(\|g(t) - S(t, g_{2n}(t))\|^2 + \|h(t) - S(t, h(t))\|^2 \right) + \gamma (\|g(t) - T(t, h(t))\|^2 + \|h(t) - S(t, g(t))\|^2) + \delta \frac{\|h(t) - S(t, g(t))\|^2}{1 + \|g(t) - S(t, g(t))\|^2 \|g(t) - T(t, h(t))\|^2}$$

$$= (\alpha + 2\gamma + \delta) \|g(t) - h(t)\|^2$$

But $0 < \alpha + 2\gamma + \delta < 1$

Hence, $g(t) = h(t)$ For all $t \in \Omega$.

Hence, S and T have a common unique fixed point in C.

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