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THIRD HANKEL DETERMINANT FOR A SUBCLASS OF ALPHA CONVEX FUNCTIONS

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Abstract: In this paper a sharp upper bound of third Hankel determinant $H_3(1)$ for the functions belonging to a subclass of alpha convex functions is established. By giving the particular values to alpha, it is easy to obtain the upper bound of $H_3(1)$ for starlike and convex functions.

Mathematics Subject classification: 30C45

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1. Introduction

Let A be the class of analytic functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1.1}$$

in the unit disc $E = \{z : |z| < 1\}$.

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By S we denote the class of functions $f(z) \in A$ and univalent in E.

M_α denote the class of functions $f(z) \in A$ and satisfying the condition

$$\operatorname{Re}\left\{\frac{zf'(z)+\alpha z^2 f''(z)}{(1-\alpha)f(z)+\alpha zf'(z)}\right\} > 0, 0 \le \alpha \le 1, z \in E. \tag{1.2}$$

The class M_α is the subclass of alpha-convex functions studied by Singh [11]. Also $M_0 \equiv S^*$, the class of starlike functions and $M_1 \equiv K$, the class of convex functions.

For the complex sequence $a_n, a_{n+1}, a_{n+2}, \ldots,$ the Hankel matrix, named after Herman Hankel(1839-1873), is the infinite matrix whose $(i, j)^{th}$ entry a_{ij} is defined by

$$a_{ij} = a_{n+i+j-2} (i, j, n \in N).$$

The q^{th} Hankel matrix ($q \in N \setminus \{1\}$) is by definition, the following q x q square sub matrix:

$$\begin{bmatrix} a_n & a_{n+1} & \ldots & a_{n+q-1} \\ a_{n+1} & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ a_{n+q-1} & \ldots & \ldots & a_{n+2q-2} \end{bmatrix}.$$

We observe that the Hankel matrix has constant positive slopping diagonals whose entries also satisfy:

$$a_{ij} = a_{i-1, j+1} (i \in N \setminus \{1\}; j \in N).$$

This also describes the Hankel matrix without reference to a particular sequence. The determinant of the q^{th} Hankel matrix, usually denoted by

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \ldots & a_{n+q-1} \\ a_{n+1} & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ a_{n+q-1} & \ldots & \ldots & a_{n+2q-2} \end{vmatrix},$$

is called the q^{th} Hankel determinant. In the particular cases

$$q = 2, n = 1, a_1 = 1 \text{ and } q = 2, n = 2,$$

the Hankel determinant simplifies respectively to

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$$H_2(1) = |a_3 - a_2^2| \quad \text{and} \quad H_2(2) = |a_2 a_4 - a_3^2|.$$

In this paper, we consider the Hankel determinant in the case $q = 3$ and $n = 1$,

$$H_3(1) = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix}.$$

$H_2(2)$ and $H_3(1)$ are respectively called second and third Hankel determinants.

For $f \in S, a_1 = 1$ so that,

$$H_3(1) = a_3(a_2 a_4 - a_3^2) - a_4(a_4 - a_2 a_3) + a_5(a_3 - a_2^2)$$

and by using the triangle inequality, we have

$$|H_3(1)| \leq |a_3||a_2 a_4 - a_3^2| + |a_4||a_2 a_3 - a_4| + |a_5||a_3 - a_2^2|. \tag{1.3}$$

Second Hankel determinant for various classes has been extensively studied by various authors including Singh[11,12], Mehrok and Singh[8] and Janteng et al.[3,4,5]. But Third Hankel determinant has been studied by some of the Researchers including Babalola [1] and Shanmugam et al.[10].

For our discussion in this paper, we consider the third Hankel determinant and obtain an upper bound to the functional $H_3(1)$ for the functions in the class M_α. Results due to Babalola [1] follows as special cases.

2. Preliminary Results

Let P be the family of all functions p analytic in E for which $\text{Re}(p(z)) > 0$ and

$$p(z) = 1 + p_1 z + p_2 z^2 + \ldots$$

for $z \in E$.

Lemma 2.1.[9] If $p \in P$, then $|p_k| \leq 2 (k = 1,2,3,\ldots)$.

Lemma 2.2.[6,7] If $p \in P$, then

$$2p_2 = p_1^2 + (4 - p_1^2)x,$$

$$4p_3 = p_1^3 + 2p_1(4 - p_1^2)x - p_1(4 - p_1^2)x^2 + 2(4 - p_1^2)(1 - |x|^2)z,$$

for some x and z satisfying $|x| \le 1, |z| \le 1$ and $p_1 \in [0,2]$.

Lemma 2.3.[2] If $p \in P$, then

$$\left| p_2 - \sigma \frac{p_1^2}{2} \right| = \begin{cases} 2(1-\sigma) & if \quad \sigma \le 0, \\ 2 & if \quad 0 \le \sigma \le 2, \\ 2(\sigma - 1) & if \quad \sigma \ge 2. \end{cases}$$

Lemma 2.4.[11] If $f(z) \in M_\alpha$, then

$$\left| a_2 a_4 - a_3^2 \right| \le \frac{1}{(1+\alpha)(1+3\alpha)}.$$

3. Main Results

Theorem 3.1 If $f \in M_\alpha$, then

$$|a_2| \le \frac{2}{1+\alpha},$$

$$|a_3| \le \frac{3}{1+2\alpha},$$

$$|a_4| \le \frac{4}{1+3\alpha}$$

and

$$|a_5| \le \frac{5}{1+4\alpha}.$$

Proof. Since $f(z) \in M_\alpha$, then there exists $p \in P$ such that

$$\frac{zf'(z) + \alpha z^2 f''(z)}{(1-\alpha) f(z) + \alpha z f'(z)} = p(z). \tag{3.1}$$

Equating coefficients in (3.1) yields

$$a_2 = \frac{p_1}{1+\alpha}, \tag{3.2}$$

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$$a_3 = \frac{p_2}{2(1+2\alpha)} + \frac{p_1^2}{2(1+2\alpha)}, \tag{3.3}$$

$$a_4 = \frac{p_3}{3(1+3\alpha)} + \frac{p_1 p_2}{2(1+3\alpha)} + \frac{p_1^3}{6(1+3\alpha)}, \tag{3.4}$$

$$a_5 = \frac{p_4}{4(1+4\alpha)} + \frac{p_1^2 p_2}{4(1+4\alpha)} + \frac{p_2^2}{8(1+4\alpha)} + \frac{p_1 p_3}{3(1+4\alpha)} + \frac{p_1^4}{24(1+4\alpha)} \tag{3.5}$$

and the results follow by triangle inequality and using Lemma 2.1.

Theorem 3.2 If $f \in M_\alpha$, then

$$|a_2 a_3 - a_4| \le \begin{cases} 2 & if \quad \alpha = 0 \\ \dfrac{2}{3(1+3\alpha)} & if \quad 0 < \alpha \le 1. \end{cases}$$

Proof. From equations (3.2),(3.3) and (3.4), we obtain

$$|a_2 a_3 - a_4| = \left| \frac{-2\alpha^2 p_1 p_2}{2(1+\alpha)(1+2\alpha)(1+3\alpha)} + \frac{(2+6\alpha-2\alpha^2)p_1^3}{6(1+\alpha)(1+2\alpha)(1+3\alpha)} - \frac{p_3}{3(1+3\alpha)} \right|. \tag{3.6}$$

Substituting for p_2 and p_3 from Lemma 2.2 and letting $p_1 = p$, we get

$$|a_2 a_3 - a_4| = \left| \frac{(1+3\alpha-4\alpha^2)p^3}{4(1+\alpha)(1+2\alpha)(1+3\alpha)} - \frac{(1+3\alpha+5\alpha^2)px(4-p^2)}{6(1+\alpha)(1+2\alpha)(1+3\alpha)} + \frac{px^2(4-p^2)}{12(1+3\alpha)} - \frac{(4-p^2)(1-|x|^2)z}{6(1+3\alpha)} \right|.$$

Since $|p| = |p_1| \le 2$ by using Lemma 2.1, we may assume that $p \in [0,2]$. Then using triangle inequality and $|z| \le 1$ with $\rho = |x|$, we obtain

$$|a_2 a_3 - a_4| \le \frac{(1+3\alpha-4\alpha^2)p^3}{4(1+\alpha)(1+2\alpha)(1+3\alpha)} + \frac{(1+3\alpha+5\alpha^2)p(4-p^2)\rho}{6(1+\alpha)(1+2\alpha)(1+3\alpha)} + \frac{(4-p^2)}{6(1+3\alpha)} + \frac{(p-2)(4-p^2)\rho^2}{12(1+3\alpha)}$$

$$= F(\rho).$$

Then

$$F'(\rho) = \frac{(1+3\alpha+5\alpha^2)p(4-p^2)}{6(1+\alpha)(1+2\alpha)(1+3\alpha)} + \frac{(p-2)(4-p^2)\rho}{6(1+3\alpha)}.$$

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Note that $F'(\rho) \geq F'(1) > 0$. Then there exists $p^* \in [0,2]$ such that $F'(\rho) > 0$ for $p \in (p^*, 2]$ and $F'(\rho) \leq 0$ otherwise. Then for $p \in (p^*, 2]$, $F(\rho) \leq F(1)$.

But

$$F(1) = \frac{(1+3\alpha+4\alpha^2)p}{(1+\alpha)(1+2\alpha)(1+3\alpha)} - \frac{2\alpha^2 p^3}{(1+\alpha)(1+2\alpha)(1+3\alpha)}$$

$$= G(p).$$

If $\alpha = 0$, we have $G(p) = p \leq 2.$

Otherwise $G(p)$ is maximum at $p = \sqrt{\frac{1+3\alpha+4\alpha^2}{6\alpha^2}}$ and is given by

$$G(p) \leq \frac{2(1+3\alpha+4\alpha^2)}{3(1+\alpha)(1+2\alpha)(1+3\alpha)}\sqrt{\frac{1+3\alpha+4\alpha^2}{6\alpha^2}}.$$

If $p \in [0, p^*]$, then $F(\rho) \leq F(0)$, that is

$$F(\rho) \leq \frac{(1+3\alpha-4\alpha^2)p^3}{4(1+\alpha)(1+2\alpha)(1+3\alpha)} + \frac{(4-p^2)}{6(1+3\alpha)}$$

$$= G(p).$$

Now $G(p)$ turns at $p = 0$ or $p = \frac{4(1+\alpha)(1+2\alpha)}{9(1+3\alpha-4\alpha^2)}$ with its maximum at $p = 0$.

Hence
$$|a_2 a_3 - a_4| \leq \frac{2}{3(1+3\alpha)}.$$

For $\alpha = 0$, Theorem 3.2 agree with the following result due to Babalola [1].

Corollary 3.2.1 If $f(z) \in S^*$, then

$$|a_2 a_3 - a_4| \leq 2.$$

For $\alpha = 1$, Theorem 3.4 gives the following result due to Babalola [1].

Corollary 3.2.2 If $f(z) \in K$, then

$$|a_2a_3 - a_4| \le \frac{1}{6}.$$

Theorem 3.3 If $f \in M_\alpha$, then

$$|a_3 - a_2^2| \le \frac{1}{1+2\alpha}.$$

Proof. Since $f(z) \in M_\alpha$, then using equations (3.2) and (3.3) we obtain

$$|a_3 - a_2^2| = \left| \frac{p_2}{2(1+2\alpha)} - \frac{p_1^2(1+2\alpha-\alpha^2)}{2(1+2\alpha)(1+\alpha)^2} \right|$$

$$= \frac{1}{2(1+2\alpha)} \left| p_2 - \frac{2(1+2\alpha-\alpha^2)}{(1+\alpha)^2} \frac{p_1^2}{2} \right|.$$

Using Lemma 2.3, with $0 \le \sigma = \dfrac{2(1+2\alpha-\alpha^2)}{(1+\alpha)^2} \le 2,$ we have

$$|a_3 - a_2^2| \le \frac{1}{1+2\alpha}.$$

Theorem 3.4 If $f \in M_\alpha$, then

$$|H_3(1)| \le \begin{cases} 16 \quad if \quad \alpha = 0 \\ \dfrac{32 + 224\alpha + 445\alpha^2 + 199\alpha^3}{3(1+\alpha)(1+2\alpha)(1+3\alpha)^2(1+4\alpha)} \quad if \quad 0 < \alpha \le 1. \end{cases}$$

.

Proof. Using Lemma 2.4 and Theorems 3.1, 3.2 and 3.3 in inequality (1.3), the above result can be easily obtained.

For $\alpha = 0$, Theorem 3.4 agree with the following result due to Babalola [1].

Corollary 3.4.1 If $f(z) \in S^*$, then

$$|H_3(1)| \le 16.$$

For $\alpha = 1$, Theorem 3.4 gives the following result due to Babalola [1].

Corollary 3.4.2 If $f(z) \in K$, then

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$$|H_3(1)| \leq \frac{15}{24}.$$

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