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ALMOST PERIODIC SOLUTION OF A MULTISPECIES DISCRETE LOTKA-VOLTERRA MUTUALISM SYSTEM

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ABSTRACT
In this paper, we consider an almost periodic multispecies discrete Lotka-Volterra mutualism system. We first obtain the permanence of the system by utilizing the theory of difference equation. By means of constructing a suitable Lyapunov function, sufficient conditions are obtained for the existence of a unique positive almost periodic solution which is uniformly asymptotically stable. An example together with numerical simulation indicates the feasibility of the main result.

General Terms
Stability and asymptotics of difference equations

Keywords
Almost periodic solution; Lotka-Volterra mutualism system; Discrete; Permanence; Uniform asymptotical stability

1. INTRODUCTION
The mutualism system has been studied by more and more scholars. Topics such as permanence, global attractivity and global stability of continuous differential mutualism system were extensively investigated(see[1-6] and the references cited therein). Xia, Cao and Cheng [1] studied a Lotka-Volterra type mutualism system with several delays

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$$\begin{cases} \dot{y}_1(t) = y_1(t)\left[r_1(t) - \sum_{i=1}^{m} a_{1i}(t)y_1(t-\tau_i(t)) + \sum_{j=1}^{n} b_{1j}(t)y_2(t-\sigma_j(t))\right], \\ \dot{y}_2(t) = y_2(t)\left[r_2(t) + \sum_{i=1}^{m} a_{2i}(t)y_1(t-\xi_i(t)) - \sum_{j=1}^{n} b_{2j}(t)y_2(t-\eta_j(t))\right]. \end{cases}$$

Some new and interesting sufficient conditions are obtained for the global existence of positive periodic solutions of the mutualism system. Their method is based on Mawhin's coincidence degree and novel estimation techniques for the a priori bounds of unknown solutions. In addition, some recent attention was on the permanence and global stability of discrete mutualism system, and many excellent results have been derived(see [7-12] and the references cited therein). Chen [8] studied a discrete mutualism model with time delays

$$\begin{cases} x_1(k+1) = x_1(k)\exp\left\{r_1(k)\left[\frac{K_1(k)+\alpha_1(k)x_2(k-\tau_2(k))}{1+x_2(k-\tau_2(k))} - x_1(k-\sigma_1(k))\right]\right\}, \\ x_2(k+1) = x_2(k)\exp\left\{r_2(k)\left[\frac{K_2(k)+\alpha_2(k)x_1(k-\tau_1(k))}{1+x_1(k-\tau_1(k))} - x_2(k-\sigma_2(k))\right]\right\}. \end{cases}$$

Sufficient conditions are obtained for the permanence of the above discrete model. Recently, as far as the discrete multispecies Lotka-Volterra ecosystem is concerned(see [11-20] and the references cited therein). Zhang et al. [12] studied an almost periodic discrete multispecies Lotka-Volterra mutualism system

$$x_i(k+1) = x_i(k)\exp\left\{a_i(k) - b_i(k)x_i(k) + \sum_{j=1, j\neq i}^{n} c_{ij}(k)\frac{x_j(k)}{d_{ij}+x_j(k)}\right\}, \quad i = 1,2,\cdots,n.$$

Sufficient conditions are obtained for the existence of a unique almost periodic solution which is globally attractive.

Specially, for the discrete two-species Lotka-Volterra mutualism system, the sufficient conditions for the existence of a unique uniformly asymptotically stable almost periodic solution are obtained. Chen [13] studied the dynamic behavior of the discrete $n + m$-species Lotka-Volerra competition predator-prey systems

$$x_i(k+1) = x_i(k)\exp\left[b_i(k) - \sum_{l=1}^{n} a_{il}(k)x_l(k) - \sum_{l=1}^{m} c_{il}(k)y_l(k)\right], \quad i = 1,2,\cdots,n,$$
$$y_j(k+1) = y_j(k)\exp\left[-r_j(k) + \sum_{l=1}^{n} d_{jl}(k)x_l(k) - \sum_{l=1}^{m} e_{jl}(k)y_l(k)\right], \quad j = 1,2,\cdots,m.$$

Sufficient conditions which ensure the permanence and the global stability of the systems are obtained; for periodic case, sufficient conditions which ensure the existence of a globally stable positive periodic solution of the systems are obtained.

In real world phenomenon, the environment varies due to the factors such as seasonal effects of weather, food supplies, mating habits, harvesting. So it is usual to assume the periodicity of parameters in the systems. However, if the various constituent components of the temporally non-uniform environment is with incommensurable (non-integral multiples) periods, then one has to consider the environment to be almost periodic since there is no a priori reason to expect the existence of periodic solutions. For this reason, the assumption of almost periodicity is more realistic, more important and more general when we consider the effects of the environmental factors. In fact, there have been many nice works on the positive almost periodic solutions of continuous and discrete dynamics model with almost periodic coefficients(see [6,11,12,21-27] and the references cited therein).

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Motivated by above, in this paper, we are concerned with the following multispecies discrete Lotka-Volterra mutualism system

$$x_i(k+1) = x_i(k)\exp\left\{a_i(k) - b_i(k)x_i(k) + \sum_{j=1,j\neq i}^{n} c_{ij}(k)\frac{x_j(k)}{d_{ij}(k)+x_j(k)}\right\}, \quad i = 1, 2, \cdots, n \tag{1.1}$$

where $\{a_i(k)\}$, $\{b_i(k)\}$, $\{c_{ij}(k)\}$ and $\{d_{ij}(k)\}$ are bounded nonnegative almost periodic sequences such that

$$\begin{aligned} 0 < a_i^l \le a_i(k) \le a_i^u, \qquad 0 < b_i^l \le b_i(k) \le b_i^u, \\ 0 < c_{ij}^l \le c_{ij}(k) \le c_{ij}^u, \qquad 0 < d_{ij}^l \le d_{ij}(k) \le d_{ij}^u, \end{aligned} \tag{1.2}$$

$i, j = 1, 2, \cdots, n,\ k \in \mathbf{Z}$. For any bounded sequence $\{f(k)\}$ defined on $\mathbf{Z}$, $f^u = \sup_{k\in\mathbf{Z}} f(k), f^l = \inf_{k\in\mathbf{Z}} f(k)$.

By the biological meaning, we will focus our discussion on the positive solutions of system (1.1). So it is assumed that the initial conditions of system (1.1) are the form:

$$x_i(0) > 0,\ i = 1, 2, \cdots, n. \tag{1.3}$$

One can easily show that the solutions of system (1.1) with the initial condition (1.3) are defined and remain positive for all $n \in N^+ = \{0, 1, 2, 3, \dots\}$.

To the best of our knowledge, this is the first paper to investigate the uniformly asymptotical stability of positive almost periodic solution of multispecies discrete Lotka-Volterra mutualism system. The aim of this paper is to obtain sufficient conditions for the existence of a unique uniformly asymptotically stable almost periodic solution of system (1.1) with initial condition (1.3), by utilizing the theory of difference equation and constructing a suitable Lyapunov function and applying the analysis technique of papers [10, 11, 21].

The remaining part of this paper is organized as follows: In Section 2, we will introduce some definitions and several useful lemmas. In the next section, we establish the permanence of system (1.1). Then, in Section 4, we establish sufficient conditions to ensure the existence of a unique positive almost periodic solution, which is uniformly asymptotically stable. The main result is illustrated by an example with a numerical simulation in the last section.

2. Preliminaries

First, we give the definitions of the terminologies involved.

Definition 2.1([28]) A sequence $x : Z \to R$ is called an almost periodic sequence if the ε-translation set of x

$$E\{\varepsilon, x\} = \{\tau \in Z : | x(n+\tau) - x(n) | < \varepsilon, \forall n \in Z\}$$

is a relatively dense set in Z for all $\varepsilon > 0$; that is, for any given $\varepsilon > 0$, there exists an integer $l(\varepsilon) > 0$ such that each interval of length $l(\varepsilon)$ contains an integer $\tau \in E\{\varepsilon, x\}$ with

$$| x(n+\tau) - x(n) | < \varepsilon, \quad \forall n \in Z.$$

τ is called an ε-translation number of $x(n)$.

Definition 2.2([29]) Let D be an open subset of R^m, $f : Z \times D \to R^m$. $f(n, x)$ is said to be almost periodic in n uniformly for $x \in D$ if for any $\varepsilon > 0$ and any compact set $S \subset D$, there exists a positive integer $l = l(\varepsilon, S)$ such that any interval of length l contains an integer τ for which

$$|f(n+\tau, x) - f(n, x)| < \varepsilon, \quad \forall (n, x) \in Z \times S.$$

τ is called an ε-translation number of $f(n, x)$.

Now, we state several lemmas which will be useful in proving our main result.

Lemma 2.1([30]) $\{x(n)\}$ is an almost periodic sequence if and only if for any integer sequence $\{k_i'\}$, there exists a subsequence $\{k_i\} \subset \{k_i'\}$ such that the sequence $\{x(n+k_i)\}$ converges uniformly for all $n \in Z$ as $i \to \infty$.

Furthermore, the limit sequence is also an almost periodic sequence.

Lemma 2.2([8]) Assume that sequence $\{x(n)\}$ satisfies $x(n) > 0$ and

$$x(n+1) \le x(n)\exp\{a(n) - b(n)x(n)\}$$

for $n \in \mathbf{N}$, where $a(n)$ and $b(n)$ are non-negative sequences bounded above and below by positive constants. Then

$$\limsup_{n\to+\infty} x(n) \le \frac{1}{b^l}\exp\{a^u - 1\}.$$

Lemma 2.3([8]) Assume that sequence $\{x(n)\}$ satisfies

$$x(n+1) \ge x(n)\exp\{a(n) - b(n)x(n)\}, \quad n \ge N_0,$$

$$\limsup_{n\to+\infty} x(n) \le x^*,$$

and $x(N_0) > 0$, where $a(n)$ and $b(n)$ are non-negative sequences bounded above and below by positive constants and $N_0 \in N$. Then

$$\liminf_{n\to+\infty} x(n) \ge \min\left\{\frac{a^l}{b^u}\exp\{a^l - b^u x^*\}, \frac{a^l}{b^u}\right\}.$$

Consider the following almost periodic difference system:

$$x(n+1) = f(n, x(n)), \quad n \in Z^+, \tag{2.1}$$

where $f : Z^+ \times S_B \to R^K, S_B = \{x \in R^k :\| x \|< B\}$, and $f(n,x)$ is almost periodic in n uniformly for $x \in S_B$ and is continuous in x. The product system of (2.1) is the following system:

$$x(n+1) = f(n, x(n)), \quad y(n+1) = f(n, y(n)), \tag{2.2}$$

and Zhang [31] obtained the following Theorem.

Theorem 2.1([31]) Suppose that there exists a Lyapunov function $V(n,x,y)$ defined for $n \in Z^+$, $\| x \|< B$, $\| y \|< B$ satisfying the following conditions:

(i) $a(\| x-y \|) \le V(n,x,y) \le b(\| x-y \|)$, where $a, b \in K$ with $K = \{a \in C(R^+, R^+) : a(0) = 0$ and a is increasing$\}$;

(ii) $\| V(n,x_1,y_1) - V(n,x_2,y_2) \|\le L[\| x_1 - x_2 \| + \| y_1 - y_2 \|]$, where $L > 0$ is a constant;

(iii) $\Delta V_{(2.2)}(n,x,y) \le -\alpha V(n,x,y)$, where $0 < \alpha < 1$ is a constant, and

$$\Delta V_{(2.2)}(n,x,y) \equiv V(n+1, f(n,x), f(n,y)) - V(n,x,y).$$

Moreover, if there exists a solution $\varphi(n)$ of (2.1) such that $\| \varphi(n) \|\le B^* < B$ for $n \in Z^+$, then there exists a unique uniformly asymptotically stable almost periodic solution $p(n)$ of (2.1) which is bounded by B^*. In particular, if $f(n,x)$ is periodic of period ω, then there exists a unique uniformly asymptotically stable periodic solution of (2.1) of period ω.

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3. Permanence

In this section, we establish a permanence result for system (1.1), which can be found by Lemma 2.2 and 2.3.

Proposition 3.1 Assume that (1.2) holds. Then any positive solution $(x_1(k), x_2(k),\ldots, x_n(k))$ of system (1.1) satisfies

$$m_i \leq \liminf_{k\to+\infty} x_i(k) \leq \limsup_{k\to+\infty} x_i(k) \leq M_i, \qquad i = 1, 2, \cdots, n,$$

where

$$M_i = \frac{1}{b_i^l}\exp\left\{a_i^u + \sum_{j=1,j\neq i}^{n} c_{ij}^u - 1\right\}, \quad m_i = \min\left\{\frac{a_i^l}{b_i^u}\exp\{a_i^l - b_i^u M_i\}, \frac{a_i^l}{b_i^u}\right\}.$$

Theorem 3.1 Assume that (1.2) holds, then system (1.1) is permanent.

According to Theorem 2.1, we first prove that there is a bounded solution of system (1.1), and then structure a suitable Lyapunov function for system (1.1).

We denote by Ω the set of all solutions $(x_1(k), x_2(k), \cdots, x_n(k))$ of system (1.1) satisfying $m_i \leq x_i(k) \leq M_i (i = 1, 2, \cdots, n)$ for all $k \in Z^+$.

Proposition 3.2 Assume that the conditions (1.2) hold. Then $\Omega \neq \Phi$.

Proof. By the almost periodicity of $\{a_i(k)\}, \{b_i(k)\}, \{c_{ij}(k)\}$ and $\{d_{ij}(k)\}$, there exists an integer valued sequence $\{\delta_p\}$ with $\delta_p \to \infty$ as $p \to \infty$ such that

$$a_i(k+\delta_p) \to a_i(k), \quad b_i(k+\delta_p) \to b_i(k), \quad c_{ij}(k+\delta_p) \to c_{ij}(k), \quad d_{ij}(k+\delta_p) \to d_{ij}(k) \quad \text{as} \quad p \to +\infty.$$

Let ε be an arbitrary small positive number. It follows from Proposition 3.1 that there exists a positive integer N_0 such that

$$m_i - \varepsilon \leq x_i(k) \leq M_i + \varepsilon, \quad k > N_0.$$

Write $x_{ip}(k) = x_i(k + \delta_p)$ for $k \geq N_0 - \delta_p$ and $p = 1, 2, \cdots$. For any positive integer q, it is easy to see that there exists a sequence $\{x_{ip}(k) : p \geq q\}$ such that the sequence $x_p(k)$ has a subsequence, denoted by $\{x_{ip}(k)\}$ again, converging on any finite interval of Z as $p \to \infty$. Thus we have a sequence $\{y_i(k)\}$ such that

$$x_{ip}(k) \to y_i(k) \text{ for } k \in Z \text{ as } p \to \infty.$$

This, combining with

$$x_i(k+1+\delta_p) = x_i(k+\delta_p)\exp\left\{a_i(k+\delta_p) - b_i(k+\delta_p)x_i(k+\delta_p) + \sum_{j=1,j\neq i}^{n} c_{ij}(k+\delta_p)\frac{x_j(k+\delta_p)}{d_{ij}(k+\delta_p) + x_j(k+\delta_p)}\right\}, \quad i = 1, 2, \cdots, n$$

gives us

$$y_i(k+1) = y_i(k)\exp\left\{a_i(k) - b_i(k)y_i(k) + \sum_{j=1,j\neq i}^{n} c_{ij}(k)\frac{y_j(k)}{d_{ij}(k) + y_j(k)}\right\}, i = 1, 2, \cdots, n.$$

We can easily see that $\{y_i(k)\}$ is a solution of system (1.1) and $m_i - \varepsilon \leq y_i(k) \leq M_i + \varepsilon$ for $k \in Z$. Since ε is an arbitrary small positive number, it follows that $m_i \leq y_i(k) \leq M_i$ and hence we complete the proof.

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4. Stability of almost periodic solution

In this section, by constructing a non-negative Lyapunov function, we will obtain sufficient conditions for uniform asymptotical stability of positive almost periodic solution of system (1.1).

Theorem 4.1 Assume that the conditions (1.2) hold, moreover, $0 < \beta < 1$, where

$$\beta = \min_{1\le i\le n}\{\beta_i\},$$
$$\beta_i = 2b_i^l m_i - b_i^{u2}M_i^2 - \sum_{j=1,j\ne i}^{n}\Big[c_{ji}^{u2} + (1+2b_i^u M_i)c_{ij}^u + (1+2b_j^u M_j)c_{ji}^u + \sum_{l=1,l\ne i,j}^{n} c_{li}^u c_{lj}^u\Big],$$

$i = 1, 2, \cdots, n$. Then there exists a unique uniformly asymptotically stable almost periodic solution $(x_1(k), x_2(k), \cdots, x_n(k))$ of system (1.1) which is bounded by Ω for all $k \in N^+$.

Proof. Let $p_i(k) = \ln x_i(k)$, $i = 1, 2, \cdots, n$. From system (1.1), we have

$$p_i(k+1) = p_i(k) + a_i(k) - b_i(k)e^{p_i(k)} + \sum_{j=1,j\ne i}^{n} c_{ij}(k)\frac{e^{p_j(k)}}{d_{ij}(k) + e^{p_j(k)}}. \tag{4.1}$$

From Proposition 3.1, we know that system (4.1) have bounded solution $(p_1(k), p_2(k), \cdots, p_n(k))$ satisfying

$$\ln m_i \le p_i(k) \le \ln M_i, \quad i = 1, 2, \cdots, n, \quad k \in Z^+.$$

Hence, $|p_i(k)| \le A_i$, where $A_i = \max\{|\ln m_i|, |\ln M_i|\}$, $i = 1, 2, \cdots, n$.

For $X \in R^n$, we define the norm $\|X\| = \sum_{i=1}^{n} |x_i|$.

Consider the product system of system (4.1)

$$\begin{cases} p_i(k+1) = p_i(k) + a_i(k) - b_i(k)e^{p_i(k)} + \sum_{j=1,j\ne i}^{n} c_{ij}(k)\frac{e^{p_j(k)}}{d_{ij}(k) + e^{p_j(k)}}, \\ q_i(k+1) = q_i(k) + a_i(k) - b_i(k)e^{q_i(k)} + \sum_{j=1,j\ne i}^{n} c_{ij}(k)\frac{e^{q_j(k)}}{d_{ij}(k) + e^{q_j(k)}}, \quad i = 1, 2, \cdots, n. \end{cases} \tag{4.2}$$

We assume that $Q = (p_1(k), p_2(k), \cdots, p_n(k))$, $W = (q_1(k), q_2(k), \cdots, q_n(k))$ are any two solutions of system (4.1) defined on $Z^+ \times S^*$; then, $\|Q\| \le B$, $\|W\| \le B$, where $B = \sum_{i=1}^{n}\{A_i + B_i\}$, $S^* = \{(p_1(k), p_2(k), \cdots, p_n(k)) | \ln m_i \le p_i(n) \le \ln M_i, i = 1, 2, \cdots, n, k \in Z^+\}$.

Let us construct a Lyapunov function defined on $Z^+ \times S^* \times S^*$ as follows:

$$V(k, Q, W) = \sum_{i=1}^{n} (p_i(k) - q_i(k))^2.$$

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It is obvious that the norm $\|Q-W\| = \sum_{i=1}^{n} |p_i(k) - q_i(k)|$ is equivalent to $\|Q-W\|_* = [\sum_{i=1}^{n} \{(p_i(k)-q_i(k))^2]^{1/2}$; that is, there are two constants $c_1 > 0, c_2 > 0$, such that

$$c_1\|Q-W\| \le \|Q-W\|_* \le c_2\|Q-W\|,$$

then,

$$(c_1\|Q-W\|)^2 \le V(k,Q,W) \le (c_2\|Q-W\|)^2.$$

Let $\psi, \varphi \in C(R^+, R^+), \psi(x) = c_1^2x^2, \varphi(x) = c_2^2x^2$; then, condition (i) of Theorem 2.1 is satisfied.

Moreover, for any $(k,Q,W), (k,\overline{Q},\overline{W}) \in Z^+ \times S^* \times S^*$, we have

$$\begin{aligned}
&|V(k,Q,W) - V(k,\overline{Q},\overline{W})| \\
&= \left|\sum_{i=1}^{n}(p_i(k)-q_i(k))^2 - \sum_{i=1}^{n}(\overline{p_i}(k)-\overline{q_i}(k))^2\right| \\
&\le \sum_{i=1}^{n}|(p_i(k)-q_i(k))^2 - (\overline{p_i}(k)-\overline{q_i}(k))^2| \\
&= \sum_{i=1}^{n}|(p_i(k)-q_i(k)) + (\overline{p_i}(k)-\overline{q_i}(k))||(p_i(k)-q_i(k)) - (\overline{p_i}(k)-\overline{q_i}(k))| \\
&\le \sum_{i=1}^{n}(|p_i(k)| + |q_i(k)| + |\overline{p_i}(k)| + |\overline{q_i}(k)|)(|p_i(k)-\overline{p_i}(k)| + |q_i(k)-\overline{q_i}(k)|) \\
&\le L[\sum_{i=1}^{n}|p_i(k)-\overline{p_i}(k)| + \sum_{i=1}^{n}|q_i(k)-\overline{q_i}(k)|] \\
&= L(\|Q-\overline{Q}\| + \|W-\overline{W}\|),
\end{aligned}$$

where $\overline{Q} = (\overline{p_1}(k), \overline{p_2}(k), \cdots, \overline{p_n}(k))$, $\overline{W} = (\overline{q_1}(k), \overline{q_2}(k), \cdots, \overline{q_n}(k))$, and $L = 4\max\{\max_{1\le i\le n}\{A_i\}, \max_{1\le i\le n}\{B_i\}\}$. Thus, condition (ii) of Theorem 2.1 is satisfied.

Finally, calculating the $\Delta V(k)$ of $V(k)$ along the solutions of system (4.2), we have

$$\begin{aligned}
&\Delta V_{(4.2)}(k) = V(k+1) - V(k) \\
&= \sum_{i=1}^{n}(p_i(k+1)-q_i(k+1))^2 - \sum_{i=1}^{n}(p_i(k)-q_i(k))^2 \\
&= \sum_{i=1}^{n}[(p_i(k+1)-q_i(k+1))^2 - (p_i(k)-q_i(k))^2] \\
&= \sum_{i=1}^{n}\left\{\left[(p_i(k)-q_i(k)) - b_i(k)(e^{p_i(k)} - e^{q_i(k)}) + \sum_{j=1,j\ne i}^{n}\frac{c_{ij}(k)d_{ij}(k)(e^{p_j(k)} - e^{q_j(k)})}{(d_{ij}(k)+e^{p_j(k)})(d_{ij}(k)+e^{q_j(k)})}\right]^2\right. \\
&\quad \left. - (p_i(k)-q_i(k))^2\right\}
\end{aligned}$$

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$$= \sum_{i=1}^{n} \Big\{ - 2b_i(k)(p_i(k) - q_i(k))(e^{p_i(k)} - e^{q_i(k)}) + b_i^2(k)(e^{p_i(k)} - e^{q_i(k)})^2$$
$$- 2b_i(k)(e^{p_i(k)} - e^{q_i(k)}) \sum_{j=1, j\neq i}^{n} \frac{c_{ij}(k)d_{ij}(k)(e^{p_j(k)} - e^{q_j(k)})}{(d_{ij}(k) + e^{p_j(k)})(d_{ij}(k) + e^{q_j(k)})}$$
$$+ \Big(\sum_{j=1, j\neq i}^{n} \frac{c_{ij}(k)d_{ij}(k)(e^{p_j(k)} - e^{q_j(k)})}{(d_{ij}(k) + e^{p_j(k)})(d_{ij}(k) + e^{q_j(k)})} \Big)^2$$
$$+ 2(p_i(k) - q_i(k)) \sum_{j=1, j\neq i}^{n} \frac{c_{ij}(k)d_{ij}(k)(e^{p_j(k)} - e^{q_j(k)})}{(d_{ij}(k) + e^{p_j(k)})(d_{ij}(k) + e^{q_j(k)})} \Big\}.$$

By the mean value theorem, it derives that

$$\mathrm{e}^{p_i(k)} - \mathrm{e}^{q_i(k)} = \xi_i(k)(p_i(k) - q_i(k)), \ i = 1, 2, \cdots, n,$$

where $\xi_i(k)$ lies between $\mathrm{e}^{p_i(k)}$ and $\mathrm{e}^{q_i(k)}$. Then, we have

$$\Delta V_{(4.2)}(k) = \sum_{i=1}^{n} \Big\{ - 2b_i(k)\xi_i(k)(p_i(k) - q_i(k))^2 + b_i^2(k)\xi_i^2(k)(p_i(k) - q_i(k))^2$$
$$- 2b_i(k)\xi_i(k)(p_i(k) - q_i(k)) \sum_{j=1, j\neq i}^{n} \frac{c_{ij}(k)d_{ij}(k)\xi_j(k)(p_j(k) - q_j(k))}{(d_{ij}(k) + e^{p_j(k)})(d_{ij}(k) + e^{q_j(k)})}$$
$$+ \Big(\sum_{j=1, j\neq i}^{n} \frac{c_{ij}(k)d_{ij}(k)\xi_j(k)(p_j(k) - q_j(k))}{(d_{ij}(k) + e^{p_j(k)})(d_{ij}(k) + e^{q_j(k)})} \Big)^2$$
$$+ 2(p_i(k) - q_i(k)) \sum_{j=1, j\neq i}^{n} \frac{c_{ij}(k)d_{ij}(k)\xi_j(k)(p_j(k) - q_j(k))}{(d_{ij}(k) + e^{p_j(k)})(d_{ij}(k) + e^{q_j(k)})} \Big\}$$
$$= \sum_{i=1}^{n} \Big\{ \Big(- 2b_i(k)\xi_i(k) + b_i^2(k)\xi_i^2(k)$$
$$+ \sum_{j=1, j\neq i}^{n} \frac{c_{ji}^2(k)d_{ji}^2(k)\xi_i^2(k)}{(d_{ji}(k) + e^{p_i(k)})^2(d_{ji}(k) + e^{q_i(k)})^2} \Big)(p_i(k) - q_i(k))^2$$
$$+2 \sum_{j=1, j\neq i}^{n} \Big(\frac{(1 - 2b_i(k)\xi_i(k))c_{ij}(k)d_{ij}(k)\xi_j(k)}{(d_{ij}(k) + e^{p_j(k)})(d_{ij}(k) + e^{q_j(k)})}$$
$$+\frac{1}{2} \sum_{l=1, l\neq i,j}^{n} \frac{c_{li}(k)d_{li}(k)\xi_i(k)}{(d_{li}(k) + e^{p_i(k)})(d_{li}(k) + e^{q_i(k)})} \frac{c_{lj}(k)d_{lj}(k)\xi_j(k)}{(d_{lj}(k) + e^{p_j(k)})(d_{lj}(k) + e^{q_j(k)})} \Big) \times$$
$$(p_i(k) - q_i(k))(p_j(k) - q_j(k)) \Big\}$$

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$$\leq \sum_{i=1}^{n}\Bigg\{\Bigg(-2b_i(k)\xi_i(k)+b_i^2(k)\xi_i^2(k) + \sum_{j=1,j\neq i}^{n}\frac{c_{ji}^2(k)d_{ji}^2(k)\xi_i^2(k)}{(d_{ji}(k)+e^{p_i(k)})^2(d_{ji}(k)+e^{q_i(k)})^2}\Bigg)(p_i(k)-q_i(k))^2 + 2\Bigg|\sum_{j=1,j\neq i}^{n}\Bigg(\frac{(1-2b_i(k)\xi_i(k))c_{ij}(k)d_{ij}(k)\xi_j(k)}{(d_{ij}(k)+e^{p_j(k)})(d_{ij}(k)+e^{q_j(k)})} + \frac{1}{2}\sum_{l=1,l\neq i,j}^{n}\frac{c_{li}(k)d_{li}(k)\xi_i(k)}{(d_{li}(k)+e^{p_i(k)})(d_{li}(k)+e^{q_i(k)})}\frac{c_{lj}(k)d_{lj}(k)\xi_j(k)}{(d_{lj}(k)+e^{p_j(k)})(d_{lj}(k)+e^{q_j(k)})}\Bigg)\times (p_i(k)-q_i(k))(p_j(k)-q_j(k))\Bigg|\Bigg\}.$$

Then, we have

$$\Delta V_{(4.2)}(k) \leq \sum_{i=1}^{n}[V_{i1}(k)+V_{i2}(k)],$$

where

$$V_{i1}(k) = \Bigg(-2b_i(k)\xi_i(k)+b_i^2(k)\xi_i^2(k)+\sum_{j=1,j\neq i}^{n}\frac{c_{ji}^2(k)d_{ji}^2(k)\xi_i^2(k)}{(d_{ji}(k)+e^{p_i(k)})^2(d_{ji}(k)+e^{q_i(k)})^2}\Bigg)(p_i(k)-q_i(k))^2$$
$$\leq \Bigg(-2b_i^l m_i + b_i^{u2}M_i^2 + \sum_{j=1,j\neq i}^{n} c_{ji}^{u2}\Bigg)(p_i(k)-q_i(k))^2,$$

$$V_{i2}(k) = 2\Bigg|\sum_{j=1,j\neq i}^{n}\Bigg(\frac{(1-2b_i(k)\xi_i(k))c_{ij}(k)d_{ij}(k)\xi_j(k)}{(d_{ij}(k)+e^{p_j(k)})(d_{ij}(k)+e^{q_j(k)})} + \frac{1}{2}\sum_{l=1,l\neq i,j}^{n}\frac{c_{li}(k)d_{li}(k)\xi_i(k)}{(d_{li}(k)+e^{p_i(k)})(d_{li}(k)+e^{q_i(k)})}\frac{c_{lj}(k)d_{lj}(k)\xi_j(k)}{(d_{lj}(k)+e^{p_j(k)})(d_{lj}(k)+e^{q_j(k)})}\Bigg)\times (p_i(k)-q_i(k))(p_j(k)-q_j(k))\Bigg|$$
$$\leq \sum_{j=1,j\neq i}^{n}\Bigg((1+2b_i^u M_i)c_{ij}^u + \frac{1}{2}\sum_{l=1,l\neq i,j}^{n} c_{li}^u c_{lj}^u\Bigg)[(p_i(k)-q_i(k))^2+(p_j(k)-q_j(k))^2]$$

Hence, we have

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$$
\begin{aligned}
\Delta V_{(4.2)}(k) \le & \sum_{i=1}^{n} \Big\{ \Big(-2b_i^l m_i + b_i^{u2} M_i^2 + \sum_{j=1, j\neq i}^{n} \Big[c_{ji}^{u2} + (1+2b_i^u M_i) c_{ij}^u + \frac{1}{2} \sum_{l=1, l\neq i,j}^{n} c_{li}^u c_{lj}^u \Big] \Big) (p_i(k) - q_i(k))^2 \\
& + \sum_{j=1, j\neq i}^{n} \Big((1+2b_i^u M_i) c_{ij}^u + \frac{1}{2} \sum_{l=1, l\neq i,j}^{n} c_{li}^u c_{lj}^u \Big) (p_j(k) - q_j(k))^2 \Big\} \\
= & \sum_{i=1}^{n} \Big\{ \Big(-2b_i^l m_i + b_i^{u2} M_i^2 + \sum_{j=1, j\neq i}^{n} \Big[c_{ji}^{u2} + (1+2b_i^u M_i) c_{ij}^u + \frac{1}{2} \sum_{l=1, l\neq i,j}^{n} c_{li}^u c_{lj}^u \Big] \Big) (p_i(k) - q_i(k))^2 \\
& + \sum_{j=1, j\neq i}^{n} \Big((1+2b_j^u M_j) c_{ji}^u + \frac{1}{2} \sum_{l=1, l\neq i,j}^{n} c_{li}^u c_{lj}^u \Big) (p_i(k) - q_i(k))^2 \Big\} \\
= & \sum_{i=1}^{n} \Big\{ \Big(-2b_i^l m_i + b_i^{u2} M_i^2 + \sum_{j=1, j\neq i}^{n} \Big[c_{ji}^{u2} + (1+2b_i^u M_i) c_{ij}^u + (1+2b_j^u M_j) c_{ji}^u \\
& + \sum_{l=1, l\neq i,j}^{n} c_{li}^u c_{lj}^u \Big] \Big) (p_i(k) - q_i(k))^2 \Big\} \\
= & -\sum_{i=1}^{n} \Big\{ \Big(2b_i^l m_i - b_i^{u2} M_i^2 - \sum_{j=1, j\neq i}^{n} \Big[c_{ji}^{u2} + (1+2b_i^u M_i) c_{ij}^u + (1+2b_j^u M_j) c_{ji}^u \\
& + \sum_{l=1, l\neq i,j}^{n} c_{li}^u c_{lj}^u \Big] \Big) (p_i(k) - q_i(k))^2 \Big\} \\
\le & -\sum_{i=1}^{n} \beta_i (p_i(k) - q_i(k))^2 \\
\le & -\beta \sum_{i=1}^{n} (p_i(k) - q_i(k))^2 \\
= & -\beta V(k, Q, W),
\end{aligned}
$$

where $\beta = \min_{1\le i\le n} \{\beta_i\}$. That is, there exists a positive constant $0 < \beta < 1$ such that

$$\Delta V_{(4.2)}(k, Q, W) \le -\beta V(k, Q, W).$$

From $0 < \beta < 1$, the condition (iii) of Theorem 2.1 is satisfied. So, according to Theorem 2.1, there exists a unique uniformly asymptotically stable almost periodic solution $(p_1(k), p_2(k), \cdots, p_n(k))$ of (4.1) which is bounded by S^* for all $k \in Z^+$. It means that there exists a unique uniformly asymptotically stable almost periodic solution $(x_1(k), x_2(k), \cdots, x_n(k))$ of (1.1) which is bounded by Ω for all $k \in Z^+$. This completed the proof.

Remark 4.1 If $n = 2$, the conditions of Theorem 4.1 can be simplified. Therefore, we have the following result.

Corollary 4.1 Let $n = 2$, and assume further that $0 < \beta < 1$, where

$$
\begin{aligned}
&\beta = \min\{\beta_{12}, \beta_{21}\}, \\
&\beta_{ij} = 2b_i^l m_i - b_i^{u2} M_i^2 - c_{ji}^{u2} - (1+2b_i^u M_i) c_{ij}^u - (1+2b_j^u M_j) c_{ji}^u,
\end{aligned}
$$

$i, j = 1, 2, j \neq i$. Then system (1.1) admits a unique uniformly asymptotically stable almost periodic solution $(x_1(k), x_2(k))$ which is bounded by Ω for all $k \in Z^+$.

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5. Example and numerical simulation

In this section, we give the following example to check the feasibility of our result.

Example Consider the following almost periodic discrete Lotka-Volterra mutualism system:

$$\begin{cases} x_1(k+1) = x_1(k)\exp\Big\{1.2 - 0.02\sin(\sqrt{2}k) - (1.05 + 0.01\sin(\sqrt{3}k))x_1(k) \\ \quad + \dfrac{(0.025 + 0.002\cos(\sqrt{5}k))x_2(k)}{0.2 + 0.003\cos(\sqrt{2}k) + x_2(k)} + \dfrac{(0.02 + 0.001\cos(\sqrt{3}k))x_3(k)}{0.4 + 0.03\cos(\sqrt{2}k) + x_3(k)}\Big\}, \\ x_2(k+1) = x_2(k)\exp\Big\{1.1 - 0.025\sin(\sqrt{3}k) - (1.08 + 0.015\sin(\sqrt{2}k))x_2(k) \\ \quad + \dfrac{(0.02 + 0.003\cos(\sqrt{2}k))x_1(k)}{0.3 + 0.02\cos(\sqrt{3}k) + x_1(k)} + \dfrac{(0.025 + 0.002\cos(\sqrt{5}k))x_3(k)}{0.2 + 0.07\sin(\sqrt{2}k) + x_3(k)}\Big\}, \\ x_3(k+1) = x_3(k)\exp\Big\{1.15 - 0.03\sin(\sqrt{2}k) - (1.1 + 0.002\sin(\sqrt{5}k))x_3(k) \\ \quad + \dfrac{(0.03 + 0.0025\cos(\sqrt{3}k))x_1(k)}{0.2 + 0.004\sin(\sqrt{2}k) + x_1(k)} + \dfrac{(0.028 + 0.0015\cos(\sqrt{2}k))x_2(k)}{0.2 + 0.004\cos(\sqrt{3}k) + x_2(k)}\Big\}. \end{cases} \tag{5.1}$$

By simple computation, we derive

$$\beta_1 \approx 0.0102, \quad \beta_2 \approx 0.0089, \quad \beta_3 \approx 0.0071.$$

Then

$$0 < \beta = \min\{\beta_1, \beta_2, \beta_3\} < 1.$$

Also it is easy to see that the conditions of Theorem 4.1 are verified. Therefore, system (5.1) has a unique positive almost periodic solution which is uniformly asymptotically stable. Our numerical simulations support our results(see Fig1-3).

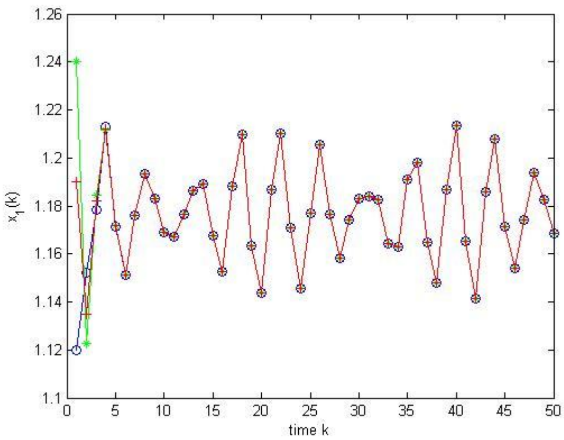

Fig 1: Dynamic behavior of $x_1(k)$ of system (5.1) with the three initial conditions (1.12,1.16,1.21), (1.24,1.09,1.12) and (1.19,0.93,0.97) for $k \in$ [1, 50], respectively.

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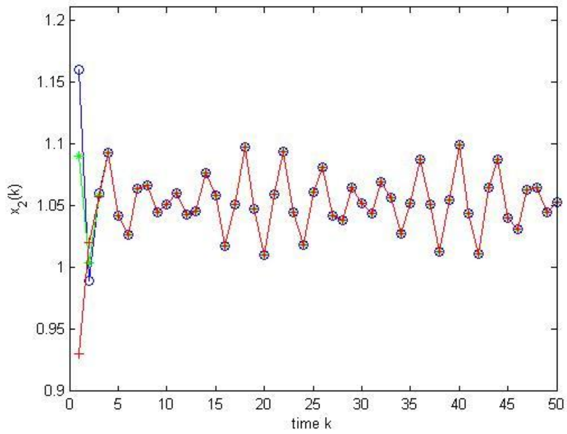

Fig 2: Dynamic behavior of $x_2(k)$ of system (5.1) with the three initial conditions (1.12,1.16,1.21), (1.24,1.09,1.12) and (1.19,0.93,0.97) for $k \in [1, 50]$, respectively.

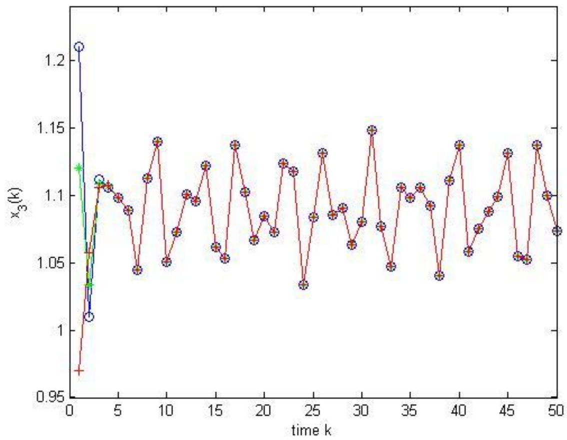

Fig 3: Dynamic behavior of $x_3(k)$ of system (5.1) with the three initial conditions (1.12,1.16,1.21), (1.24,1.09,1.12) and (1.19,0.93,0.97) for $k \in [1, 50]$, respectively.

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