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EIGENVALUES AND EIGENFUNCTIONS OF MIXED INTEGRAL EQUATION USING KREIN'S METHOD

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Abstract

In this paper, the existence of a unique solution of mixed integral equation (**MIE**) of the first kind is considered in the space $L_2[-1,\ 1]\times C(0,T)$, T< 1. The integral term of position, in $L_2[-1,\ 1]$, has a discontinuous kernel, while the integral term of time, in $C(0,T)$, has a continuous kernel. Using a numerical, we have system of Fredholm integral equations (**SFIEs**) of the first find. Then, using Krein's method, the solution of the integral system is obtained in the form of spectral relationships (**SRs**) of eigenvalues and eigenfunctions. Many special cases are considered and many applications in fluid mechanics and contact problems are discussed.

Keywords and Phrases

Mixed integral equation, system of Fredholm integral equations, Krein's method, spectral relationships, Chebyshev polynomials (**CPs**).

MSC: (2010): 45B05, 45R10.

1. INTRODUCTION

Singular integral equations of the first kind have received considerable interest in the mathematical literatures, because of their many field of applications in different areas of sciences, for example see [1- 4]. The solution of these IEs can be obtained analytically using one of the following methods: Cauchy method [5], potential theory method [6], orthogonal polynomials method [7], integral transformation methods [4-7] and Krein's method [8]. More information for the spectral relationships and its using in mathematical physics problems can be found in [9-14].

Consider the MIE of the first kind

$$\int_0^t\int_{-a}^{a} F(t,\tau)k\left|\frac{x-y}{\lambda}\right|\Phi(y,\tau)dyd\tau=\pi f(x,t)=\pi\left[r(t)-f_*(x)\right]$$

$$k(z)=\int_0^{\infty}\frac{L(v)\mathrm{os}v\ z}{v}dv,\quad L(v)=\frac{m+v}{1+v},\quad m\geq 1. \tag{1.1}$$

$$\left((x,y)\in[-1,1];\ (t,\tau)\in[0,\mathrm{T}],\ \mathrm{T}\leq 1;\ \lambda\in(0,\infty)\right).$$

Under the dynamic condition

$$\int_{-a}^{a}\Phi(x,t)dx=\mathrm{P}(t) \tag{1.2}$$

The function $L(v)$ is continuous and positive for $\mathbf{v}\in(0,\infty)$ and satisfies the following asymptotic equalities

$$L(v)=m-(m-1)v+O(v^3),\quad v\to 0,$$

$$L(v)=1-\frac{m-1}{v}+O\left(\frac{1}{v^3}\right),(v\to\infty,\quad m\geq 1). \tag{1.3}$$

The MIE (1.1), under the condition (1.2), can be investigated from the contact problem of a rigid surface (G,υ) having an elastic material occupying the domain $[-a,a]$, where $f_*(\mathrm{x})$ is describing the surface base of a stamp. This stamp is impressed into an elastic layer surface by a variable known force $\mathrm{P}(t)$, $t\in[0,\mathrm{T}]$, $\mathrm{T}<1$, whose eccentricity of application $e(t)$, that case a rigid displacement $\gamma(t)$. Here, $\mathbf{G}$ is called the displacement magnitude and υ is Poisson's coefficient see [11, 12].

In order to guarantee the existence of unique solution of (1.1), we assume, for the two constants E and D, the following conditions:

(i) The kernel of position satisfies $\left\{\int_{-a}^{a}\int_{-a}^{a}k^2\left|\frac{x-y}{\lambda}\right|dxdy\right\}^{\frac{1}{2}}=E.$

(ii) The positive continuous kernel, which represents the resistance force of the material, $F(t,\tau)\in C([0,\mathrm{T}]\times[0,\mathrm{T}])$ and satisfies $F(t,\tau)<D.$

(iii) The continuous function of time $\gamma(t)\in C[0,\mathrm{T}]$, while the position function $f_*(\mathrm{x})\in L_2[-a,a]$ and $f(x,t)\in L_2[-a,a]\times C[0,\mathrm{T}].$

(iv) The unknown potential function $\Phi(x,t)$ satisfies Hölder condition with respect to time and Lipschitz condition with respect to position.

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In the remainder part of this work, we use a quadratic numerical method in (1.1) to obtain linear SFIEs of the first kind. Then, using Krein's method, with the aid of Chebyshev polynomials of the first and second kind with its properties, the solution of SFIEs can be obtained in the form of eigenvalues and eigenfunctions. The eigenfunctions, in this work, take the form of Chebyshev polynomials of the first and second kind. Many special cases are derived and discussed from the work. Moreover, some applications in contact problems and fluid mechanics are considered.

2. System of FIEs. If we divide the interval [0,T], $0 \le t \le \mathrm{T} \le 1$ as $0 \le t_0 < t_1 < \cdots < t_N = \mathrm{T}$, when $t = t_k,\ k = 0, 1, 2, \ldots, \ell$. The MIE (1.1) takes the form, see [2]

$$\int_0^t \int_{-a}^{a} F(t,\tau) k(x,y) \varphi(y,\tau) dy d\tau = \sum_{j=0}^{k} u_j F_{j,k} \int_{-a}^{a} k(x,y) \phi_j(y) dy = \pi f_k(x). \tag{2.1}$$

In (2.1) we neglect the error term, $O\left(\hbar_\ell^{p+1}\right)$ where $\hbar_\ell = \max h_j$, $h_j = t_{j+1} - t_j$. The constant u_j defined as the characteristic number, see [2]. Also we used the following notations

$$\Phi(x,t_\ell) = \Phi_\ell(x), F(t_\ell, t_j) = F_{\ell,j}, f(x,t_\ell) = f_\ell(x).$$

The boundary condition (1.2), becomes

$$\int_{-a}^{a} \Phi_k(x) dx = \mathrm{P}_k \qquad (\mathrm{P}_k \text{ are constants }), \tag{2.2}$$

Let, in the kernel of (1.1), $m = 1$ and $\lambda \to \infty$, such that the term $(x - y)$ is very small , then using the famous relation [7]

$$\int_0^\infty \frac{\cos v\ z}{v} dv = -\ln z + d \qquad (d \text{ is a constant }). \tag{2.3}$$

The formula (2.1), with the aid of (2.3), becomes

$$\sum_{j=0}^{\ell} u_j F_{j,\ell} \int_{-a}^{a} \ln \frac{1}{|x - y|} \Phi_j(y) dy = \pi g_k(x), \quad \left(g_k(x) = f_k(x) - \frac{\mathrm{P}_\ell}{\pi} \sum_{j=0}^{\ell} u_j F_{j,\ell}\right). \tag{2.4}$$

The formula (2.4) represents linear SFIEs of the first kind with logarithmic kernel. To

obtain the solution, we use Krein's method, see [8, 9].

2. PRINCIPAL OF KREIN'S METHOD

The corresponding solution of (2.4), under the condition (2.2), using the principal Krein's method for even and odd function, respectively, is given as

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$$u_j F_{j,\ell}\Phi_j^{+}(x)=\frac{\mathrm{I}(a)}{\pi\,[\ln\frac{2}{a}+d\,]}\cdot\frac{1}{\sqrt{a^2-x^2}}-\frac{2}{\pi^2}\int_x^a\frac{du}{\sqrt{u^2-x^2}}\frac{d}{du}[u\ \frac{d}{du}\ \int_0^u\frac{g_\ell^{+}(y)dy}{\sqrt{u^2-y^2}}]\ .(2.5)$$

and

$$u_j\ F_{j,\ell}\ \Phi_j^{-}(x)=\frac{-2}{\pi^2}\frac{d}{dx}\int_x^a\frac{u\ du}{\sqrt{u^2-x^2}}\int_0^a\frac{dg_\ell^{-}(y)}{\sqrt{u^2-y^2}}.. \qquad (2.6)$$

Here, in (2.5), (2.6) we define

$$\mathrm{I}(u)=\frac{2}{\pi}\left[\int_0^u\frac{g_\ell^{+}(y)\,dy}{\sqrt{u^2-y^2}}+u\ln(\frac{2}{u}+d)\frac{d}{du}\int_0^u\frac{g_\ell^{+}(y)\,dy}{\sqrt{u^2-y^2}}\right]. \qquad (2.7)$$

and

$$g_\ell(x)=g_\ell^{+}(x)+g_\ell^{-}(x),\quad g_\ell^{+}(-x)=\pm g_\ell^{\pm}(x);\quad \Phi_j(x)=\Phi_j^{+}(x)+\Phi_j^{-}(x),$$

$$\Phi_\ell^{+}(-x)=\pm\,\Phi_\ell^{\pm}(x);\quad (x\in(-a,a),\,j=1,\ 2,\dots,\ell,\ell=1,\ 2,\ \dots,\ \mathrm{N}). \qquad (2.8)$$

3. METHOD OF SOLUTION

To obtain the solution of (2.5) and (2.6) we state the following theorem

Theorem 1: The spectral relationships for the SFIEs (2.4) with logarithmic kernels, under the conditions (2.3), take the form

$$\sum_{j=0}^{\ell}u_j F_{j,\ell}\int_{-a}^{a}\frac{[-\ln(x-y)+d\,]}{\sqrt{a^2-y^2}}\mathrm{T}_{n_j}(\tfrac{y}{a})dy=\begin{cases}\pi\,\mathrm{P}_\ell\left[\ln(\tfrac{2}{a})+d\right] & n_\ell=0\\ \pi\,\mathrm{T}_{n_\ell}(\tfrac{x}{a}),\ \ \ell=1,\ 2,\dots,\mathrm{N}. & n_\ell\geq 1\end{cases} \qquad (3.1)$$

where $\mathrm{T}_{n_j}(x)$, j = 1, 2, …, ℓ are the Chebyshev polynomials of the first find of order n .

Proof: The proof of (3.1) depends on the following lemmas

Lemma 1: For all positive integers n_j, $\mathbf{a}=1$, we have

$$\mathrm{I}_{n_j}(u)=\ 2\left[\frac{1}{2}\,\mathrm{P}_{n_j}^{(-1,\,0)}\left(2u^2-1\right)+\ n_j\ u^2\ln\left(\tfrac{2}{u}\ +\ d\ \right).\mathrm{P}_{n_j-1}^{(0,1)}\left(2u^2-1\right)\right] \qquad (3.2)$$

where $\mathrm{P}_{n_j}^{(\alpha,\,\beta)}(x)$ are Jacobi polynomials.

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Proof: to prove (3.2) let $g_{\ell}^{+}(\mathrm{y})=\mathrm{T}_{2n_\ell}(y);\ \mathrm{T}_{2n_\ell}(\mathrm{x})$ are the Chebyshev polynomials of the first find, then (2.7) can be written in the form

$$\mathrm{I}_{n_j}(u)=\frac{2}{\pi}\left[D_{n_\ell}(u)+u\ \ln\left(\frac{2}{u}+d\right)\frac{d}{du}D_{n_\ell}(u)\right],\quad D_{n_\ell}(u)=\int_0^{\mathrm{u}}\frac{\mathrm{T}_{2n_\ell}(s)ds}{\sqrt{u^2-s^2}}. \tag{3.3}$$

Using the substation $\mathbf{s}=\mathbf{ut}$ and the relation $\mathrm{T}_{2n_j}(\mathrm{x})=\mathrm{T}_{n_j}(2x^2-1)$, we have

$$D_{n_\ell}=\int_0^1(1-t^2)^{\frac{-1}{2}}\,\mathrm{T}_{n_\ell}(2t^2u^2-1)dt. \tag{3.4}$$

Using the famous relation between Chebyshev polynomials $T_{n_j}(x)$, Legendre polynomials $\mathrm{P}_{n_j}(x)$ and Jacobi polynomials $\mathrm{P}_{n_j}^{(\alpha,\beta)}(x)$, see [15]

$$\int_{-1}^{1}(1-t^2)^{-1/2}\,\mathrm{T}_{n_j}(1-t^2y)dt=\frac{\pi}{2}\left[\mathrm{P}_{n_j}(1-y)+\mathrm{P}_{(n_j-1)}(1-y)\right], \tag{3.5}$$

$$2\,\mathrm{P}_{n_j}^{(-1,0)}(x)=\mathrm{P}_{n_j}(x)-\mathrm{P}_{(n_j-1)}(x),$$

the formula (3.5), yields

$$D_{n_\ell}(u)=\frac{\pi}{2}\,\mathrm{P}_{n_\ell}^{(-1,0)}(2u^2-1) \tag{3.6}$$

Also, the first derivative of (3.6) takes the form

$$\frac{dD_{n_\ell}(\mathrm{u})}{d\ u}=n_\ell\pi u\ \mathrm{P}_{(n_\ell-1)}^{(0,1)}(2u^2-1),\quad n_\ell=1,2,\ldots;\ell,\ (\ell=1,2,\ldots,\mathrm{N}) \tag{3.7}$$

($\mathrm{P}_{n_\ell}^{(\alpha,\beta)}(x)=0$ for negative integer) .

Finally, introducing (3.6), (3.7) in (3.3) , we obtain the required result.■

Corollary1:Putting, in (3.3), $u=1$ and then using the relation $\mathrm{P}_n^{(\alpha,\beta)}(1)=\frac{\Gamma(n+\alpha+1)}{n!\Gamma(1+\alpha)}$, we have

$$D_{n_\ell}(1)=2n_\ell\ln(2+d),\quad (\Gamma(.)\text{is the Gamma function}). \tag{3.8}$$

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Corollary2: The value of the second derivatives $\frac{d}{du}\left(u\frac{dD_{n_\ell}}{du}\right)$ is given by

$$\frac{d}{du}\left(u\frac{dD_{n_\ell}}{du}\right) = D_{n_\ell}^{(2)}(u) = 2n_\ell \pi u\left[\mathrm{P}_{n_\ell-1}^{(0,1)}\left(2u^2-1\right)+\left(n_\ell+1\right)u^2\mathrm{P}_{n_\ell-2}^{(1,2)}\left(2u^2-1\right)\right]. \tag{3.9}$$

Lemma 2: The value of the following integral

$$\mathrm{A}_{n_\ell}(x)=\int_x^1 \frac{du}{\sqrt{u^2-x^2}}\frac{d}{du}\left[u\frac{d}{du}\int_0^u \frac{\mathrm{T}_{2n_\ell}(s)ds}{\sqrt{u^2-s^2}}\right], \tag{3.10}$$

takes the form

$$\mathrm{A}_{n_\ell}(x)=\frac{\pi^{\frac{3}{2}}n_\ell!}{\sqrt{2}\Gamma\left(n_\ell-\frac{1}{2}\right)}\cdot\frac{1}{\sqrt{1-y}}\left[\frac{1-y}{2n_\ell-1}\mathrm{P}_{(n_\ell-1)}^{(\frac{1}{2},\frac{1}{2})}(y)-(1+y)\mathrm{P}_{n_\ell-1}^{\left(\lambda-\frac{1}{2},\frac{3}{2}\right)}(y)\right]+\frac{\sqrt{2}n_\ell\pi}{\sqrt{1-y}}. \tag{3.11}$$

$$(y=2x^2-1; n_\ell=1,2,\ldots;\ell=1,2,\ldots,\mathrm{N})$$

Proof: For proving the lemma, we rewrite (3.10) in the form

$$\mathrm{A}_{n_\ell}(x)=2n_\ell\pi\left[\int_x^1\frac{u\,\mathrm{P}_{n_\ell-1}^{(0,1)}\left(2u^2-1\right)du}{\sqrt{u^2-x^2}}+\left(n_\ell+1\right)\int_x^1\frac{u^3\mathrm{P}_{n_\ell-2}^{(1,2)}\left(2u^2-1\right)du}{\sqrt{u^2-x^2}}\right]. \tag{3.12}$$

Using, in (3.12), the substitution $2u^2-1=y,\ 2x^2-1=z$, to have

$$A_{n_\ell}(x)=A_{n_\ell}(z)=\frac{\pi n_\ell}{\sqrt{2}}\int_z^1\frac{\mathrm{P}_{(n_\ell-1)}^{(0,1)}(y)dy}{\sqrt{y-z}}+\frac{\pi n_\ell\left(n_\ell+1\right)}{2\sqrt{2}}\int_z^1\frac{y\,\mathrm{P}_{(n_\ell-2)}(y)dy}{\sqrt{y-z}}+\frac{\pi n_\ell\left(n_\ell+1\right)}{2\sqrt{2}}\int_z^1\frac{\mathrm{P}_{n_\ell-2}^{(1,2)}(y)dy}{\sqrt{y-z}}, \tag{3.13}$$

If we put $y=1-(1-z)v$, then (3.13) yields

$$A_{n_\ell}=\frac{\pi n_\ell}{\sqrt{2}}\sqrt{1-z}\int_0^1(1-v)^{\frac{-1}{2}}\mathrm{P}_{n_\ell-1}^{(0,1)}\left[1-(1-z)v\right]dv$$

$$+\frac{\pi n_\ell\left(n_\ell+1\right)}{2\sqrt{2}}\sqrt{1-z}\,(1+z)\int_0^1(1-v)^{\frac{-1}{2}}\mathrm{P}_{n_\ell-2}^{(1,2)}\left[1-(1-z)v\right]dv$$

$$+\frac{n_\ell\left(n_\ell+1\right)\pi}{2\sqrt{2}}(1-z)^{\frac{3}{2}}\int_0^1(1-v)^{\frac{1}{2}}\mathrm{P}_{n_\ell-2}^{(1,2)}\left[1-(1-z)v\right]dv\ . \tag{3.14}$$

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If we use the famous formulas [15]

$$\int_0^1 z^{\lambda-1}(1-z)^{r-1} \mathrm{P}_n^{(\alpha,\beta)}(1-\gamma z)dz = \frac{\Gamma(\alpha+n+1)\Gamma(\lambda)\Gamma(r)}{n!\Gamma(1+\alpha)\Gamma(\lambda+r)} {}_3F_2(-n,n+\alpha+\beta+1;\lambda,\alpha+1,\lambda+r;\frac{\gamma}{2}) \quad (3.15)$$

and

$$\mathrm{P}_n^{(\alpha,\beta)}(v) = \binom{n+\alpha}{n} F\left(-n, n+\alpha+\beta+1; \frac{1-v}{2}\right) \quad (3.16)$$

Here, ${}_3F_2(\alpha_1,\alpha_2,\alpha_3;\beta_1,\beta_2;z);(R_e\lambda>0,R_e r>0)$, is the generalized hypergeometric series and $F(\alpha,\beta;\gamma;z)$ is the hypergeometric Gauss function, the first integral term of (3.14) becomes

$$\int_0^1 (1-v)^{\frac{-1}{2}} \mathrm{P}_{n_\ell-1}^{(0,1)}[1-(1-z)v]dv = \frac{\sqrt{\pi}(n_\ell-1)!}{\Gamma\left(n_\ell+\frac{1}{2}\right)} \mathrm{P}_{n_\ell-1}^{\left(\frac{1}{2},\frac{1}{2}\right)}(z) \quad (3.17)$$

Also, using the same way, the second and third integral term of (3.14), yield

$$\int_0^1 (1-v)^{\frac{-1}{2}} \mathrm{P}_{n_\ell-2}^{(1,2)}[1-(1-z)v]dv = \frac{1}{1-z}\left\{\frac{-2\sqrt{\pi}\Gamma(n_\ell)}{(n_\ell+1)\Gamma\left(n_\ell-\frac{1}{2}\right)} \mathrm{P}_{n_\ell-1}^{\left(\frac{-1}{2},\frac{3}{2}\right)}(z) + \frac{2}{n_\ell+1}\right\} \quad (3.18)$$

and

$$\int_0^1 (1-v)^{\frac{1}{2}} \mathrm{P}_{n_\ell-2}^{(1,2)}[1-(1-z)v]dv = \frac{-\sqrt{\pi}(n_\ell-1)!\mathrm{P}_{n_\ell-1}^{\left(\frac{1}{2},\frac{1}{2}\right)}(z)}{(n_\ell+1)(1-z)\Gamma\left(n_\ell+\frac{1}{2}\right)} + \frac{2}{(n_\ell+1)(1-z)} \quad (3.19)$$

Introducing the three formulas (3.17) - (3.19) in (3.13) the lemma is proved.■

Finally, to prove the theorem, we write (3.11) in the Chebyshev polynomials form, for this purpose, we must consider the following famous formulas, see [15, 16]

(i) Relation between Jacobi and Gegenbauer polynomials

1- $\mathrm{P}_n^{\left(\frac{-1}{2},\frac{-1}{2}\right)}(2x^2-1) = \frac{\Gamma\left(n+\frac{1}{2}\right)\Gamma(\lambda)}{\sqrt{\pi}\Gamma(n+\lambda)} C_{2n}^{\lambda(x)}$, 2- $\mathrm{P}_n^{\left(\lambda-\frac{1}{2},\frac{1}{2}\right)}(2x^2-1) = \frac{\Gamma\left(n+\frac{3}{2}\right)\Gamma(\lambda)}{\sqrt{\pi}x\Gamma(n+\lambda+1)} C_{2n+1}^{\lambda}(x)$.

(ii) Relation between Chebyshev and Gegenbauer polynomials

3. $\lim_{\lambda\to 0}\Gamma(\lambda)C_n^{\lambda}(x) = \frac{2}{n}\mathrm{T}_n(x); \quad (n\geq 1)$.

Using these famous relations in (3.11), one has

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$$A_{n_\ell}(x) = \frac{n_\ell \pi \left(1 - \mathrm{T}_{2n_\ell}(x)\right)}{\sqrt{1-x^2}}, (n \geq 1). \quad , \tag{3.20}$$

Introducing (3.20) and (3.3) in (2.5) , the theorem is proved.■

By using the same way, we can prove this theorem

Theorem 2: The spectral relationships for the **SFIEs** with the kernel defined by (2.3) and the known function is odd is given by

$$\sum_{J=0}^{\ell} u_j F_{j,\ell} \int_{-1}^{1} \left[\ln \frac{1}{|x-s|} + d \right] \mathrm{T}_{2n_j-1}(s)\, ds = \frac{\pi}{2n_\ell - 1} \mathrm{T}_{2n_\ell - 1}(x); \quad \left(n_\ell \geq 1, \ell = 1, 2, \ldots, \mathrm{N}\right) \tag{3.21}$$

The proof of theorem 2 can be obtained directly by following the same way of theorem 1.■

4. CONCLUSION AND RESULTS

From the above results and discussion, the following may be concluded

(1) The contact problem of a rigid surface of an elastic material, when a stamp of length $2a$ is impressed into an elastic layer surface of a strip by a variable $\mathrm{P}(t), 0 \leq t \leq \mathrm{T} < 1$, whose eccentricity of application $e(t)$,represents MIE of the first kind.

(2) The quadratic numerical method used transforms the **MIE** in position and time into **SFIEs** in position. Moreover, the **SFIEs** depend on the number of derivatives of $F(t,\tau)$ with respect to time $t, t \in [0, \mathrm{T}], \mathrm{T} < 1$.

(3) The displacement problems of ant plane deformation of an infinite rigid strip with width $2a$, putting on an elastic layer of thickness h is considered as a special case of this work when $t = 1$, $F(t,\tau) = 1$, $f(x,t) = \mathrm{H}$ and $\Phi(x,1) = \Psi(x)$. Here H represents the displacement magnitude and $\Psi(x)$ the unknown function represents the displacement stress.

(4) The problems of infinite rigid strip with width $2a$ impressed in a viscous liquid layer of thickness h , when the strip has a velocity resulting from the impulsive force $v = v_0 e^{-iwt}, i = \sqrt{-1}$, where v_0 is the constant velocity , w is the angular velocity resulting rotating the strip about z-axis are considered as special case of this work , when $F(t,\tau) =$ constant , and $t = 1$ see [4] .

(5)In the discussion (3) and (4) , when $h \to \infty$, this means the depth of the liquid (Fluid mechanics) or the thickness of elastic material (contact problem) becomes an infinite .

(6)The three kinds of the displacement problem, in the theory of elasticity and mixed contact problems, which discussed in [4,7] ,are considered special cases of this work .

(7) Many important relationships can be derived from (3.1)

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If $n_j = 2m_j$, $\frac{x}{a} = \frac{\sin\frac{\xi}{2}}{\sin\frac{\alpha}{2}}$, $\frac{y}{a} = \frac{\sin\frac{\eta}{2}}{\sin\frac{\alpha}{2}}$. And if $n_j = 2m_j + 1$, $\frac{x}{a} = \frac{\tan\frac{\xi}{2}}{\tan\frac{\alpha}{2}}$, $\frac{y}{a} = \frac{\tan\frac{\eta}{2}}{\tan\frac{\alpha}{2}}$, we have the following **SFIEs**

$$\sum_{j=0}^{\ell} u_j F_{j,\ell} \int_{-\alpha}^{\alpha} \left[\ln \frac{1}{2\left|\sin\frac{\xi-\eta}{2}\right|} + d \right] \psi_j(\xi) d\xi = h_k(\eta) \tag{4.1}$$

The above system leads to the following **SRs**

$$\sum_{j=0}^{\ell} u_j F_{j,\ell} \int_{-\alpha}^{\alpha} \left[\ln \frac{1}{2\left|\sin\frac{\xi-\eta}{2}\right|} + d \right] \frac{\mathrm{T}_{2n_j}\left(\frac{\sin\frac{\eta}{2}}{\sin\frac{\alpha}{2}}\right)\cos\left(\frac{\eta}{2}\right)d\eta}{\sqrt{2(\cos\eta - \cos\alpha)}} = \begin{cases} \pi \mathrm{P}_\ell\left(\ln\frac{2}{\sin\alpha} + d\right) & m_\ell = 0 \\ \frac{\pi}{2m_\ell}\mathrm{T}_{m_\ell}\left(\frac{\sin\frac{\xi}{2}}{\sin\frac{\alpha}{2}}\right) & m_\ell \ge 1, \ell = 1,2,\ldots,\mathrm{N} \end{cases} \tag{4.2}$$

and

$$\sum_{j=0}^{\ell} u_j F_{j,\ell} \int_{-a}^{a} \left(\ln \frac{1}{2\left|\sin\frac{\xi-\eta}{2}\right|} \right) \frac{\mathrm{T}_{2m_j+1}\left(\frac{\tan\frac{\eta}{2}}{\tan\frac{\alpha}{2}}\right)\cos\frac{\eta}{2}d\eta}{\sqrt{2(\cos\eta - \cos\alpha)}} = \frac{\pi}{2m_\ell + 1}\mathrm{T}_{2m_j+1}\left(\frac{\tan\frac{\xi}{2}}{\tan\frac{\alpha}{2}}\right) \quad (m_\ell \ge 0) \tag{4.3}$$

(ii) Differentiating (3.1) with respect to x, we have

$$\sum_{j=0}^{\ell} u_j F_{j,\ell} \int_{-a}^{a} \frac{\mathrm{T}_{n_j}\left(\frac{y}{a}\right)}{y-x} \frac{dy}{\sqrt{a^2 - y^2}} = \pi U_{n_\ell - 1}\left(\frac{x}{a}\right) \qquad n_\ell \ge 1$$

$$\sum_{j=0}^{\ell} u_j F_{j,\ell} \int_{-a}^{a} \frac{dy}{(y-x)\sqrt{a^2 - y^2}} = 0 \tag{4.4}$$

where $U_{n_\ell}\left(\frac{x}{a}\right)$ are the Chebyshev polynomials of the second kind.

Also (4.4) yields

$$\sum_{j=0}^{\ell} u_j F_{j,\ell} \int_{-\alpha}^{\alpha} \cot\frac{\eta-\xi}{2} \frac{\mathrm{T}_{n_j}\left(\frac{\tan\frac{\eta}{2}}{\tan\frac{\alpha}{2}}\right)}{\sqrt{2(\cos\eta - \cos\alpha)}} \cos\left(\frac{\eta}{2}\right) d\eta$$

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$$= \begin{cases} 0 & n_\ell = 0 \\ 2\operatorname{cosec}\left(\frac{\alpha}{2}\right) U_{2m_\ell - 1}\left(\dfrac{\tan\frac{\xi}{2}}{\tan\frac{\alpha}{2}}\right) & n_\ell = 2m_\ell \\ 2\left[\operatorname{cosec}\left(\frac{\alpha}{2}\right) U_{2m_j - 1}\left(\dfrac{\tan\frac{\xi}{2}}{\tan\frac{\alpha}{2}}\right) + (-1)^2 \dfrac{\sin\alpha}{1+\cos\alpha}\left[\tan\frac{\alpha}{4}\right]^{2m_\ell - 2}\right] & n_\ell = 2m_\ell - 1 \end{cases}, \tag{4.5}$$

$$\frac{1}{2}\sum_{j=0}^{\ell} u_j F_{j,\ell} \int_{-a}^{a} \frac{\sec\frac{\eta}{2}.\cot\frac{\eta-\xi}{2}}{\sqrt{2(\cos\eta - \cos\alpha)}} \mathrm{T}_{n_j}\left(\frac{\tan\frac{\eta}{2}}{\tan\frac{\alpha}{2}}\right) d\eta = \begin{cases} \operatorname{cosec}\left(\frac{\alpha}{2}\right)\sec^2\left(\frac{\xi}{2}\right) U_{n_\ell - 1}\left(\dfrac{\tan\frac{\xi}{2}}{\tan\frac{\alpha}{2}}\right) & n \geq 1 \\ \sec\left(\frac{\alpha}{2}\right).\tan\left(\frac{\xi}{2}\right) & n_\ell = 0 \end{cases} \tag{4.6}$$

(8) The mixed integral equation with Carleman kernel can be established from this work by using the following relation

$$\ln|x-y| = h(x,y)|x-y|^{-\upsilon}; \quad 0<\upsilon<1, \tag{4.7}$$

where $h(x,y) = |x-y|^{\upsilon} \ln|x-y|$ is a smooth function. The importance of Carleman kernel came from the work of Arutiunion [17] who has shown that, the contact problem of nonlinear theory of plasticity, in its first approximation reduce to **FIE** of the first kind with Carleman kernel .

(9) The relation between the eigenvalues n and the corresponding Chebyshev polynomial T_n are obtained in the following figures

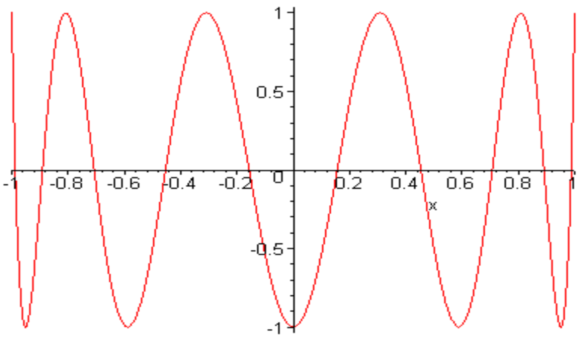

Fig. 1 n=5

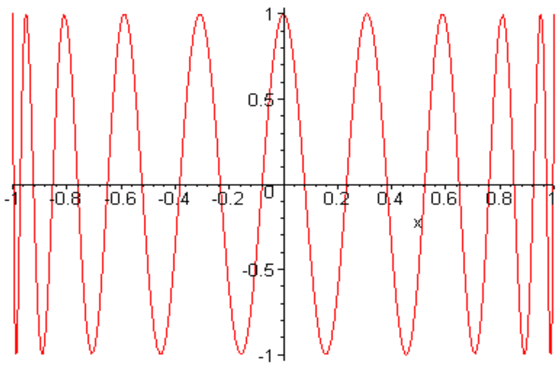

Fig.2: *n*=10

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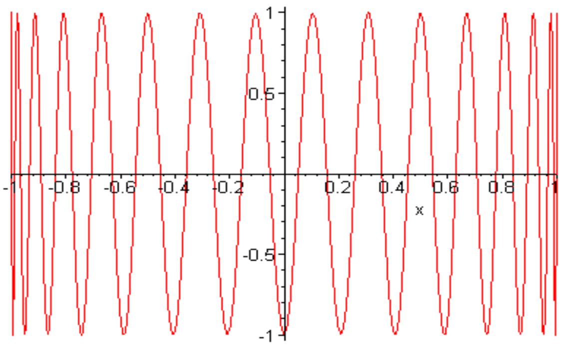

Fig.3: ***n*=30**

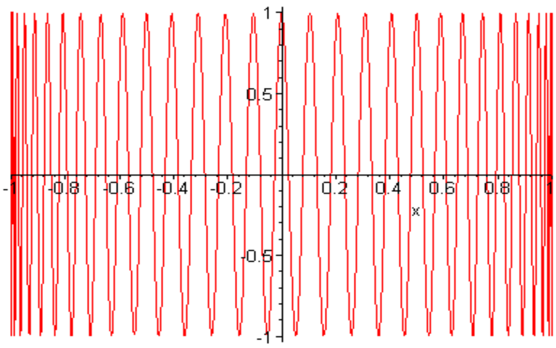

Fig.4.6: *n*=15

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