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SEARCHING SOLITON SOLUTIONS TO THE BURGER-HUXLEY AND THE KLEIN GORDON EQUATIONS

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ABSTRACT

In this article, we execute an extension of the (G'/G) method viz. the $(G'/G,1/G)$ method to construct exact traveling wave solutions to nonlinear evolution equations. We attain kink, singular kink, periodic, singular periodic and other types of solitons entangled free parameters. Choosing particular values of the parameters, different type of solitons are derived from the traveling waves. The efficiency of the $(G'/G,1/G)$-expansion method is shown by applying this method to the Burgers-Huxley and the Klein-Gordon equations. The illustrated method appears to be easier and more convenient due to using symbolic computation system Maple software.

Keywords: The Burger-Huxley equation; the Klein-Gordon equation; nonlinear evolution equations; solitary traveling wave; Kink, computer algebra system.

1. INTRODUCTION

The laws of nature inherently are nonlinear [20]. Partial differential equations are the basis of all physical theorems and formulate basic laws of nature. Indeed, most of the complex phenomena such as motion, conservation, reaction, diffusion, equilibrium and so on can be modeled by PDEs. Due to its essence, PDEs are studied extensively by specialist and practitioners in science and engineering. Generally, these studies pave the way into many entries throughout the scientific literature and reflect a development of mathematical theories and analytical techniques to solve PDEs with the explanation of the underlying governed phenomena.

In the early days of nonlinear science, since computers were not available, diverse attempts were made to solve the nonlinear PDE for exact solution e.g. to reduce the PDEs to ODE and solved by hand calculators. Scientific computation has emerged over the past few years as the most versatile tool to complement theory and experiments for solving PDEs. The advent of modern computational technologies the scenario has since changed dramatically. The nonlinear PDE can now be solved effectively by means of sophisticated computers, with due attention to the accuracy of the solutions. When exact solution exists, their existence greatly assists in the understanding of the original nonlinear PDE. These solutions also help in the quantitative estimation of how the solutions of certain classes' problems evolve in time.

As there is no panacea for all types of disease, there is no single method which is central to solve the nonlinear PDEs. Rather there are separate theories which are applied for each of the major types of PDE that commonly arise, as nonlinear PDEs modeled the rich variety of physical, geometric and probabilistic phenomena in different aspects. So, research focuses on various particular methods for seeking exact solutions that are important for applications within and outside of mathematics, with the hope that these solutions will reflect the insight of PDE and clues of their origins.

In recent years, many authors have investigated the exact solution of nonlinear evolution equations. Some common methods are available in most of the textbooks an many powerful and effective methods have also been presented in different papers by several authors. The inverse scattering method is used to solve the Cauchy problem for integrable partial differential equations. The other methods for exact solution of integrable PDEs are used such as the Hirota method, the truncated Painleve expansion method, the exp-function method, the Backlund transform method and so on [20], [27].

Besides integrable equations, there are many nonlinear evolution equations which are partially integrable equations because they turn out to be integrable for some values of their parameters. Like as integrable PDEs, there also numerous methods which are used for looking the exact solutions of non integrable PDEs. The most remarkable methods that are used to find explicit solutions of these types of PDEs are the Weierstrass elliptic function method, the Jacobi elliptic function method, the tanh-function method etc [27]. Many asymptotic methods have been suggested for the boundary value problems such as the foremost Adomain decomposition method which was introduced and developed by George Adomain, in 1994, the variational iteration method, which was proposed by Ji-Huan He, in 1999, and the homotopy perturbation method and later developed of these methods by many researchers [27]. Beside theses, up to now, many more powerful methods have been proposed and established to obtain analytical solutions of NLEEs, such as the Wronskian determinant technique, homogeneous balance method, the Darboux transformation, the Lie symmetries method, the theta function method, the complex hyperbolic function method, the F-expansion method, the sub-ODE method, symmetry method, first integral method, the trial function method, the nonlinear transform method, sine-cosine method, tanh-coth method [20], [27].

In recent years, due to the availability of symbolic computation packages such as Maple or Mathematica that enables us to perform the daunting and complex computation on computer, direct methods to construct exact solutions of nonlinear PDEs have become more and more attractive. In 2008, Wang et al. first proposed such $\left(G'/G\right)$- expansion method, which is one of the most efficient direct method to construct travelling wave solutions of NLEES [25].

Since then, this (G'/G) extension method applied to many NLEEs and successfully obtained new exact solutions. Many researchers also worked on this (G'/G) -expansion method to improve and enhance this expansion method [9-10]. Different modification and extensions have also been proposed by severals e.g. Zhang et al. [28], extended the method to deal with evolution equations with variable coefficients. A remarkable work was also done by Zhang [29], to some special nonlinear equations where the balance numbers are not integer. Akbar et al. modified as well as applied this method and derived abundant traveling wave solutions of different celebrated NLEEs [1-5].

In 2010, Li et al. firstly introduced the (G′/G, 1/G)-expansion method which can be thought of as an extension of the (G′/G)-expansion method [26]. They applied this extension method to the Zakharov equations and the traveling wave solutions are successfully obtained. The principle idea of this expansion method is that the exact traveling wave solutions of NLEEs can be expressed in the form a polynomial in two variables (G'/G) and $(1/G)$ where $G=G\left(\xi\right)$ satisfies a second order linear ordinary differential equation (LODE). Recently, Demiray et al. have also applied this method to the Boussinesq type equations for searching the exact solutions [7-8]. The $(G'/G,1/G)$ -method is also applied to various nonlinear PDEs such as Phi-Four equation, Boussinesq type equations, Gardner-KP equation [6-8], for analytical solutions. Very recently, Akbar et al. applied both the (G'/G) and $(G'/G,1/G)$ expansion methods to the various nonlinear PDEs [13-16]. However, still this method is of little use to different nonlinear evolution equations due to an extra variable $(1/G)$, compared to (G'/G) expansion method.

In this work, we have applied two variables $(G'/G,1/G)$ -expansion method to the Burgers-Huxley and the Klein-Gordon nonlinear evolution equations to check the effectiveness of this method and to obtain further new form exact traveling wave solutions.

We now briefly describe how this paper is arranged. In Section 2, we define the concept of $(G'/G,1/G)$ -expansion method briefly. In Sections 3 we have applied this extension method to the Burgers-Huxley and the Klein-Gordon nonlinear evolution. Graphical representations of the obtaining results are presented in section 4. Results and discussion are given in section 5. Concluding remarks are given in section 6.

2. DESCRIPTION OF $(G'/G,1/G)$-EXPANSION METHOD

In this section, we describe the main steps of the $(G'/G,1/G)$-expansion method succinctly. Before going to discuss the main steps of this expansion method to find travelling wave solution of nonlinear evolution equations, some interpretations are required. Li et al. is the pioneer of this method [1], which is summarized with the following remarks:
At first, consider a second order liner ordinary differential equation (LODE)

$$\frac{d^2G(\xi)}{d\xi^2}+\lambda G(\xi)=\mu \tag{1}$$

and we let

$$\phi = G'/G, \quad \psi = 1/G \tag{2}$$

for simplicity herein and after. Using the relation $\frac{d^2G(\xi)}{d\xi^2}=-\lambda G(\xi)-\mu$ from equation (1) with addition to Eq. (2), yields

$$\phi' = -\phi^2+\mu\psi-\lambda, \quad \psi' = -\phi\psi \tag{3}$$

The general solution of the ODE (1) depends upon the value of the parameter whether $\lambda<0$, $\lambda>0$ or $\lambda=0$.

Case1: If $\lambda<0$, the general solution of the LODE (1) is given as,

$$G(\xi)=A_1\sinh(\sqrt{-\lambda}\xi)+A_2\cosh(\sqrt{-\lambda}\xi)+\frac{\mu}{\lambda}$$

and it can be easily derived the relation

$$\psi^2=\frac{-\lambda}{\lambda^2\sigma+\mu^2}(\phi^2-2\mu\psi+\lambda) \tag{4}$$

where A_1 and A_2 are two arbitrary constants and $\sigma=A_1^2-A_2^2$.

Case2: If $\lambda>0$, the general solution of Eq. (1) has the form

$$G(\xi)=A_1\sin(\sqrt{\lambda}\xi)+A_2\cos(\sqrt{\lambda}\xi)+\frac{\mu}{\lambda}$$

and hence it can be easily found that

$$\psi^2=\frac{\lambda}{\lambda^2\rho-\mu^2}(\phi^2-2\mu\psi+\lambda) \tag{5}$$

where A_1 and A_2 are two arbitrary constants and $\rho=A_1^2+A_2^2$.

Case3: When $\lambda=0$, then the general solution of the Eq. (1) is of the form

$$G(\xi)=\frac{\mu}{2}\xi^2+\zeta_1\xi+\zeta_2$$

and by some calculations, it can be deduced

$$\psi^2=\frac{\lambda}{A_1^2-2\mu A_2}(\phi^2-2\mu\psi) \tag{6}$$

where A_1 and A_2 are two arbitrary constants.

Suppose that we have the following nonlinear evolution partial differential equation,

$$F(u,u_t,u_x,u_y,u_{tt},u_{xt},u_{xx}, ...)=0 \qquad (7)$$

Mathematically, the left-hand side of Eq. (7) is a polynomial in $u=u(x,t)$ and its various partial derivatives, in which the highest order derivatives and the nonlinear terms, are involved. In order to solve Eq. (7) using $(G'/G,1/G)$-expansion method the following steps are to be considered.

Step 1: Introducing the travelling wave variable $u(x,t)=u(\xi)$, $\xi = x-ct$, where c is a constant, that allows us to reduce the equation (7) to an ODE for $u=u(\xi)$ in the form

$$P(u,u',u'', ...)=0 \qquad (8)$$

where $u'=\dfrac{du}{d\xi},\ u''=\dfrac{d^2u}{d\xi^2},\ldots$ and so on. Next, integrate the Eq. (8) as many times as possible and set the arbitrary constants of integration to be zero for simplicity.

Step 2: Suppose that the general solution of the ordinary differential equation (8) can be expressed by a polynomial in ϕ and ψ as,

$$u(\xi)=\sum_{i=0}^{m} a_i\phi^i+\sum_{i=1}^{m} b_i\phi^{i-1}\psi \qquad (9)$$

where $\phi = G'/G$, $\psi = 1/G$ and $G = G(\xi)$ satisfies the second order LODE (1), $a_i (i = 0,1,2,...,m), b_i (i = 1,2,...,m), c, \lambda$ and μ are arbitrary constants which will be determined in later. Moreover, the positive integer m can be determined by using the condition homogenous balance between the highest order derivatives and the nonlinear terms appearing in Eq. (8).

Step 3: As $u(\xi)$ is a solution of Eq. (8), substituting (9) into (8), using (3) and (4) (or using (3), (5) and (3), (6)) the left side of Eq. (8) transforms into a polynomial in ϕ and ψ , wherein the degree of ψ is not larger than 1. Vanishing each coefficient of the polynomial in ϕ,ψ to zero yields a system of algebraic equations in $a_i (i = 0,1,2,...,m),\ b_i (i=1,2,...,m),\ c,\ \lambda,\ \mu,\ A_1$ and A_2.

Step 4: Solving the algebraic solutions in Step 3 with the help of mathematical graphing utility Maple or Mathematica and substituting the obtained values of $a_i (i = 0,1,2,...,m),\ b_i (i=1,2,...,m),\ c,\lambda,\mu, A_1$ and A_2 into (9), one can finally obtain the travelling wave solutions expressed by the hyperbolic functions of Eq. (8) (or can be expressed by trigonometric function if $\lambda > 0$ and rational function if $\lambda = 0$).

3. APPLICATION OF THE $(G'/G,1/G)$-EXPANSION METHOD

3.1 Burgers-Huxley equation

The Burgers–Huxley equation describes a wide class of physical nonlinear phenomena, e.g. a prototype model for describing the interaction between reaction mechanisms, convection effects and diffusion transports. The Burgers-Huxley equation reads as:

$$u_t - u_{xx} = uu_x + u(k-u)(u-1) . \qquad (10)$$

This nonlinear PDE has a lot of applications in many fields such as physics, economics and ecology, metallurgy, combustion, mathematics and engineering. Upon using the wave variable transformation $u(x,t)=u(\xi),\ \xi = x-ct$, the Burgers-Huxley equation can be converted to the ODE as:

$$cu' + uu' + u'' + u(k-u)(1-u) = 0. \tag{11}$$

Now, balancing the nonlinear term u^3 with the highest order derivative u'' in Eq. (11) gives $m=1$. Therefore, the highest value of m in the polynomial (9) is 1. Substituting $m=1$, the solution formula (9) becomes

$$u(\xi) = a_0 + a_1\phi + b_1\psi \tag{12}$$

Case 1: When $\lambda < 0$

Substituting the value of $u(\xi)$ and its various derivatives into Eq. (11), using (3) and (4), the Eq. (11) becomes a polynomial in ϕ and ψ. Equating each coefficient of equal to zero, a system of algebraic equations is found as follows:

$$\phi^3 : -\lambda^2\sigma a_1^2 + 2\lambda^2\sigma a_1 - \mu^2 a_1^2 + \lambda b_1^2 + 2\mu^2 a_1$$

$$\phi^2\psi : -2\lambda^2\sigma a_1 b_1 + 2\lambda^2\sigma b_1 - 2\mu^2 a_1 b_1 + 2\mu^2 b_1$$

$$\phi^2 : -c\lambda^2\sigma a_1 - \lambda^2\sigma a_0 a_1 - \lambda^2\sigma a_1^2 - c\mu^2 a_1 - \lambda\mu a_1 b_1 - \mu^2 a_0 a_1 - \mu^2 a_1^2 + \lambda\mu b_1 + \lambda b_1^2$$

$$\cdots$$

$$\psi : c\lambda^2\mu\sigma a_1 - \lambda^3\sigma a_1 b_1 + \lambda^2\mu\sigma a_0 a_1 + c\mu^3 a_1 + \lambda^3\sigma b_1 - 2\lambda^2\sigma a_0 b_1 + \lambda\mu^2 a_1 b_1 + \mu^3 a_0 a_1 + \lambda^2\sigma b_1 - \lambda\mu^2 b_1 - 2\lambda\sigma\mu b_1^2 - 2\mu^2 a_0 b_1 + \mu^2 b_1$$

$$\phi^0 : c\lambda^3\sigma a - \lambda^3\sigma a_0 a_1 - c\lambda\mu^2 a_1 - \lambda^2\mu a_1 b_1 - \lambda^2\sigma a_0^2 - \lambda\mu^2 a_0 a_1 + \lambda^2\mu b_1 + \lambda^2\sigma a_0 + \lambda^2 b_1^2 - \mu^2 a_0^2 + \mu^2 a_0$$

Solving the above system of algebraic equations with the help of Maple, we get the solution set of the Burgers-Fisher equation as

$$a_0 = \frac{1}{2},\ a_1 = -1, b_1 = \frac{1}{2}\sqrt{16\mu^2 + \sigma}, c = -2k + \frac{1}{2}, \lambda = -\frac{1}{4} \tag{13}$$

Inserting, $\phi = G'/G$ and $\psi = 1/G$ into Eq. (12), we achieve

$$u(\xi) = \frac{\sqrt{-\lambda}(A_1\cosh(\sqrt{-\lambda}\xi) + A_2\sinh(\sqrt{-\lambda}\xi))}{A_1\sinh(\sqrt{-\lambda}\xi) + A_2\cosh(\sqrt{-\lambda}\xi) + \frac{\mu}{\lambda}} + \frac{b_1}{A_1\sinh(\sqrt{-\lambda}\xi) + A_2\cosh(\sqrt{-\lambda}\xi) + \frac{\mu}{\lambda}} + b_0 \tag{14}$$

Now substituting (13) into (14), we get

$$u(\xi) = \frac{\frac{1}{2}\left(A_1\cosh\left(\frac{\xi}{2}\right) + A_2\sinh\left(\frac{\xi}{2}\right)\right)}{A_1\sinh\left(\frac{\xi}{2}\right) + A_2\cosh\left(\frac{\xi}{2}\right) - 4\mu} + \frac{\frac{1}{2}(\sqrt{16\mu^2 + \sigma})}{A_1\sinh\left(\frac{\xi}{2}\right) + A_2\cosh\left(\frac{\xi}{2}\right) - 4\mu} + \frac{1}{2} \tag{15}$$

If we consider the particular values as $A_1 = 0,\ A_2 = \sqrt{-\sigma}$ and $\mu = 0,$ we obtain the kink traveling wave solution as:

$$u(\xi) = \frac{1}{2}\left[\tanh\left(\frac{\xi}{2}\right) + i\operatorname{sec}h\left(\frac{\xi}{2}\right) + 1\right] \tag{16}$$

Finally, substituting $\xi = x - ct$, we get the traveling wave kink type solution of the Burger-Huxley equation (10)

$$u(x,t) = \frac{1}{2}\left[\tanh\left(\frac{1}{2}(x-ct)\right) + i\operatorname{sec}h\left(\frac{1}{2}(x-ct)\right) + 1\right]. \tag{17}$$

Similarly, again setting the particular values as $A_1 = \sqrt{\sigma},\ A_2 = 0$ and $\mu = 0,$ we obtain the traveling wave solution as:

$$u(x,t) = \frac{1}{2}\left[\coth\left(\frac{1}{2}(x-ct)\right) + \operatorname{csc}h\left(\frac{1}{2}(x-ct)\right) + 1\right] \tag{18}$$

Case 2: When $\lambda > 0$

Similar to the *case1*, substituting the values of $u(\xi)$ and its various derivatives in (11), alongside Eqs. (3) and (5), the left hand side of (11), becomes a polynomial equation in ϕ and ψ. Equating each coefficient equal to zero, a system of algebraic equations is found and solving them with the help of Maple, we get the solution set (for brevity and simplicity algebraic equations are not given)

$$a_0 = \frac{1}{2},\ a_1 = -1, b_1 = \frac{1}{2}\sqrt{16\mu^2 - \rho}, c = -2k + \frac{1}{2}, \lambda = -\frac{1}{4} \tag{19}$$

Again, substituting these values into Eq.(12), along with $\phi = (G'/G)$ and $\psi = (1/G)$, we get

$$u(\xi) = \frac{\frac{1}{2}\left(A_1\cos\left(\frac{\xi}{2}\right) + A_2\sin\left(\frac{\xi}{2}\right)\right)}{A_1\sin\left(\frac{\xi}{2}\right) + A_2\cos\left(\frac{\xi}{2}\right) + 4\mu} + \frac{\frac{1}{2}(\sqrt{16\mu^2 - \rho})}{A_1\sin\left(\frac{\xi}{2}\right) + A_2\cos\left(\frac{\xi}{2}\right) + 4\mu} + \frac{1}{2} \tag{20}$$

Plugging in the particular values as $A_1 = 0,\ A_2 = \sqrt{\rho}$ and $\mu = 0,$ we obtain the periodic traveling wave solution as:

$$u(x,t) = \frac{1}{2}\left[\tan\left(\frac{1}{2}(x-ct)\right) + i\sec\left(\frac{1}{2}(x-ct)\right) + 1\right] \tag{21}$$

Similarly, setting the particular values as $A_1 = \sqrt{\rho},\ A_2 = 0$ and $\mu = 0,$ we extract the following wave solution as:

$$u(x,t) = \frac{1}{2}\left[\cot\left(\frac{1}{2}(x-ct)\right) + i\operatorname{cos}ec\left(\frac{1}{2}(x-ct)\right) + 1\right]. \tag{22}$$

Case 3: When $\lambda = 0$ (Rational function solutions)

In the case of $\lambda = 0$, the Burgers-Huxley equation provides the trivial solution as:

$$a_0 = 0, a_1 = 0, b_1 = 0, \mu = \mu \tag{23}$$

3.2 Application to the Klein-Gordon equation

The Klein-Gordon equation reads as

$$u_{tt} - u_{xx} + u - u^2 = 0 \tag{24}$$

Now, substituting the wave variable $u = u(\xi),\ \xi = x - ct$ with its various partial derivatives in the nonlinear PDE, (24), and integrating , we get the ODE

$$(c^2 - 1)u'' + u - u^2 = 0 \tag{25}$$

Balancing the nonlinear term u^2 with higher-order derivative u'', we get

$$m = 2 \tag{26}$$

On substituting the value of m into (9), the ansatz solution is of the form

$$u(\xi) = a_0 + a_1\phi + a_2\phi^2 + b_1\psi + b_2\phi\psi \tag{27}$$

where the coefficients a_0, a_1, a_2, b_1, b_2 are arbitrary constants which are to be determined later. The type of the solutions of (25) depends on the character of λ.

Case1: when $\lambda < 0$ (Hyperbolic function solutions)

Substituting the value of $u(\xi)$ from (27) and its various derivative along with (3) and (5), the left hand side of Eq. (25) becomes a polynomial in ϕ and ψ. Equating each coefficient equal to zero, we derive a set of algebraic equations for $a_0, a_1, a_2, b_1, b_2, \lambda$ and μ and solving the system of algebraic equations with the support of Maple, we obtain (for succinctness algebraic equations are not given)

Solving the system of equations by using Maple software, we obtain the following results

$$a_0 = -2,\ a_1 = 0, a_2 = -\frac{3}{\lambda}, b_1 = \frac{3\mu}{\lambda}, b_2 = \frac{3\sqrt{-(\lambda^2\sigma + \mu^2)}}{\lambda^{3/2}}, c = \sqrt{\frac{1-\lambda}{\lambda}} \tag{28}$$

Substituting these values in (27)

$$\begin{aligned} u(\xi) = &-\frac{3\left(A_1\cosh\left(\sqrt{-\lambda}\xi\right)\sqrt{-\lambda} + A_2\sinh\left(\sqrt{-\lambda}\xi\right)\sqrt{-\lambda}\right)^2}{\left(A_1\sinh\left(\sqrt{-\lambda}\xi\right) + A_2\cosh\left(\sqrt{-\lambda}\xi\right) + \frac{\mu}{\lambda}\right)^2\lambda} \\ &+ \frac{3\left(A_1\cosh\left(\sqrt{-\lambda}\xi\right)\sqrt{-\lambda} + A_2\sinh\left(\sqrt{-\lambda}\xi\right)\sqrt{-\lambda}\right)\left(\sqrt{-\left(\lambda^2\sigma + \mu^2\right)}\right)}{\left(A_1\sinh\left(\sqrt{-\lambda}\xi\right) + A_2\cosh\left(\sqrt{-\lambda}\xi\right) + \frac{\mu}{\lambda}\right)^2\sqrt[3]{\lambda}} \\ &+ \frac{3\mu}{\left(A_1\sinh\left(\sqrt{-\lambda}\xi\right) + A_2\cosh\left(\sqrt{-\lambda}\xi\right) + \frac{\mu}{\lambda}\right)\lambda} - 2 \end{aligned} \tag{29}$$

In particular case, when $A_1 = 0,\ A_2 = \sqrt{-\sigma}$ and $\mu = 0$, we get the solution

$$u(x,t) = 3\tanh^2(\sqrt{-\lambda}\,\xi) + 3i\tanh(\sqrt{-\lambda}\,\xi)\operatorname{sec}h(\sqrt{-\lambda}\,\xi) - 2 \tag{30}$$

Substituting $\xi = x - ct$

$$u(x,t) = 3\tanh^2(\sqrt{-\lambda}\,(x-ct)) + 3i\tanh(\sqrt{-\lambda}\,(x-ct))\operatorname{sec}h(\sqrt{-\lambda}\,(x-ct)) - 2 \tag{31}$$

Also for the case, when $A_1 = \sqrt{\sigma},\ A_2 = 0$ and $\mu = 0$, we get the solution

$$u(x,t) = 3\coth^2(\sqrt{-\lambda}\,(x-ct)) + 3\coth(\sqrt{-\lambda}\,(x-ct))\operatorname{csc}h(\sqrt{-\lambda}\,(x-ct)) - 2 \tag{32}$$

Case2: when $\lambda > 0$ (trigonometric function solutions)

Following the previous scheme, again substituting the value of $u(\xi)$ from (27) and its various derivative along with (3) and (4), the left hand side of Eq. (25) becomes a polynomial in ϕ and ψ. Setting each coefficient of this polynomial to zero, we derive a set of algebraic equations for $a_0, a_1, a_2, b_1, b_2, \lambda, \mu$ and the solution set becomes:

$$a_0=-2,\ a_1=0, a_2=-\frac{3}{\lambda}, b_1=\frac{3\mu}{\lambda}, b_2=\frac{3\sqrt{(\lambda^2\sigma-\mu^2)}}{\lambda^{3/2}}, c=\sqrt{\frac{1-\lambda}{\lambda}} \tag{33}$$

Inserting these solutions in(27), we get

$$\begin{aligned} u(\xi)= & -\frac{3\left(A_1\cos\left(\sqrt{\lambda}\xi\right)\sqrt{\lambda}-A_2\sin\left(\sqrt{\lambda}\xi\right)\sqrt{\lambda}\right)^2}{\left(A_1\sin\left(\sqrt{\lambda}\xi\right)+A_2\cos\left(\sqrt{\lambda}\xi\right)+\frac{\mu}{\lambda}\right)^2\lambda} \\ & +\frac{3\left(A_1\cos\left(\sqrt{\lambda}\xi\right)\sqrt{\lambda}-A_2\sin\left(\sqrt{\lambda}\xi\right)\sqrt{\lambda}\right)\left(\sqrt{(\lambda^2\sigma-\mu^2)}\right)}{\left(A_1\sin\left(\sqrt{\lambda}\xi\right)+A_2\cos\left(\sqrt{\lambda}\xi\right)+\frac{\mu}{\lambda}\right)^2\sqrt[3]{\lambda}} \\ & +\frac{3\mu}{\left(A_1\sin\left(\sqrt{\lambda}\xi\right)+A_2\cos\left(\sqrt{\lambda}\xi\right)+\frac{\mu}{\lambda}\right)\lambda}-2 \end{aligned} \tag{34}$$

In special case, when $A_1=0,\ A_2=\sqrt{\sigma}$ and $\mu=0$, we get the following periodic solution

$$u(x,t)=-3\tan^2(\sqrt{\lambda}\,\xi)-3\tan(\sqrt{\lambda}\,\xi)\sec(\sqrt{\lambda}\,\xi)-2 \tag{35}$$

Putting $\xi=x-ct$, we get

$$u(x,t)=-3\tan^2(\sqrt{\lambda}\,(x-ct))-3\tan(\sqrt{\lambda}\,(x-ct))\sec(\sqrt{\lambda}\,(x-ct))-2 \tag{36}$$

Again choosing the parameter $A_1=\sqrt{\sigma},\ A_2=0$ and $\mu=0$, we get another periodic solution

$$u(x,t)=-3\cot^2(\sqrt{\lambda}\,(x-ct))+3\cot(\sqrt{\lambda}\,(x-ct))\csc(\sqrt{\lambda}\,(x-ct))-2 \tag{37}$$

In the case of $\lambda=0$, the $(G'/G,1/G)$-expansion provides only trivial solution and consequently no physically significant exact rational solution can be obtained of the Klein-Gordon equation.

4. GRAPHICAL REPRESENTATION OF THE OBTAINING RESULTS

In this section, we discuss the graphical explanation of the exact traveling wave solutions which are derived by the $(G'/G,1/G)$ -expansion method, particularly, to the exact solutions of the Burger-Huxley equation in the earlier section. For conciseness, we have only plotted the graph of Eq. (17), (18) and (21) of the exact solution of the Burger-Huxley equation in the figure (1), (2) and (3) respectively.

Taking, the modulus of Eq. (17), we attain

$$|u(x,t)|=2(1+\tanh x) \tag{38}$$

The traveling wave solution in the form $\tanh x$ is called the kink solution. Kink waves are traveling waves which rise or descent from one asymptotic state to another state. From the Fig.1, it is evident that the graph rises or descent from one asymptotical state at $\xi\to-\infty$ to another asymptotical state at $\xi\to+\infty$. The graph is translated upward version of fundamental $\tanh x$ function. The physical structure of the graph in Fig.1 depends mainly on the sign of the wave speed c whether $c>0$ or $c<0$. We observe from Eq. (17) that, the wave disturbance moving in the positive x-direction for the value of $c>0$, and the wave travels in the negative x-direction if $c<0$.

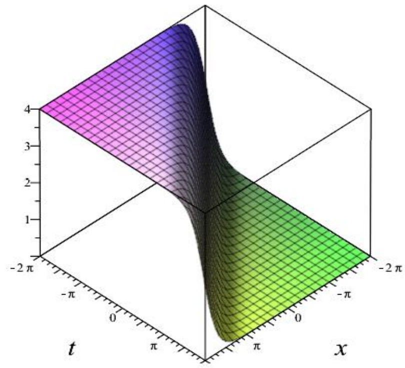

Figure 1. Graph of the absolute value of the solution Eq. (17) in the domain $-2\pi \le x, t \le 2\pi$

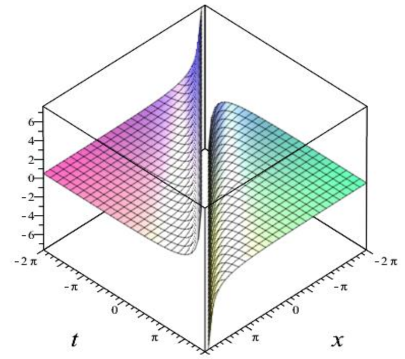

Figure 2. Graph of the solution Eq. (18) in the domain $-2\pi \le x, t \le 2\pi$

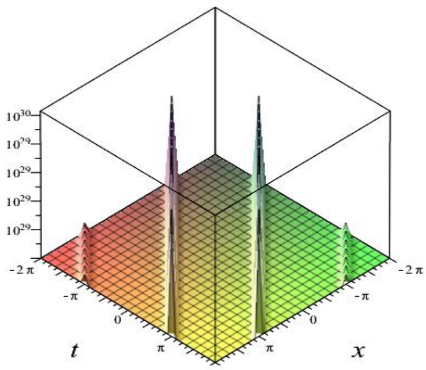

Figure 3. Graph of the absolute value of the solution Eq. (21) in the domain $-2\pi \le x, t \le 2\pi$

The Fig.2 is depicted in the domain $-2\pi \le x, t \le 2\pi$. Interesting feature of this traveling wave is that in one side the wave asymptotically tends to $+\infty$ along a line in x-space, whereas asymptotically tends to $-\infty$ in other side along the same line. This is due to the fact that the hyperbolic function both $\coth$ and $\cos ech$ are undefined at 0. The Fig.3 shows the periodic nature of the solution of Burger-Huxley equation.

5. RESULTS AND DISCUSSIONS

In this section, we discuss the $(G'/G, 1/G)$ -expansion method. In delving the $(G'/G, 1/G)$ -expansion method, we provide an overview of (G'/G) method. The (G'/G) -method is based on the assumptions that the solution of the nonlinear evolution equations can be expressed by a polynomial in (G'/G), where $G = G(\xi)$ satisfies the second order linear ordinary differential equation (LODE) $G''(\xi)+\lambda G(\xi)+\mu G(\xi)=0$ where λ and μ are arbitrary constants. Similar to the (G'/G)-method, the principle idea of $(G'/G,1/G)$-expansion method is that the exact traveling wave solutions of NLEEs can be expressed by a polynomial in two variables (G'/G) and $(1/G)$ where $G=G(\xi)$ satisfies a second order linear ordinary differential equation (LODE). The degree of the polynomial can be determined by considering the homogeneous balance between the nonlinear term and highest order derivatives appear in a given NLEE. Besides, the coefficients of the polynomial can be determined by solving a set of algebraic equations equating to zero resulting from the process of using the method. This method is based on explicit linearization of NLEEs and the traveling wave variable transformation $u(x,t)=u(\xi),\ \xi=x-ct$ transforms the PDE into ODE. When $\mu=0$ in Eq. (1) and $b_i=0$ in Eq.(9), the $(G'/G,1/G)$ expansion method transforms into (G'/G)-expansion method. Moreover, if we take the special values of two parameters A_1 and A_2, familiar type of kink wave solutions are originated from the general solution.

Many techniques are available to searching the analytical solution of NLEEs and each of them have some advantages and disadvantages. Some methods provide solutions in a series form. In this case burning question arises to investigate the convergence of approximation series. For instance, Adomian decomposition method, variational iteration method [27], depends only on the initial conditions and converges to the exact solution of the problem. Some methods need linearization or to convert the inhomogenous boundary conditions to homogeneous, and so on [20]. In addition to, all numerical methods e.g. finite difference or finite element methods it is necessary to must have boundary and initial conditions. The main advantages of $(G'/G,1/G)$-expansion method over other methods are it can be applied directly without using linearization, perturbation or any other restrictive assumption that may change the physical behavior of the model under discussion. It is worthy to note out that if the order of the reduced ODE is higher comparatively, it is mostly not possible to find out a useful solution. In spite of this limitation, this method is useful for finding new exact solutions that are important in different field contexts and to validate the numerical solutions.

6. CONCLUSION

In this article, we have implemented the $(G'/G,1/G)$-expansion method and tested the effectiveness through the Burger-Huxley and Klein-Gordon equations. As there is no unified rule to solve all types or even a class of nonlinear evolution equations, it is seen that the exact traveling wave solutions can be obtained of the Burger-Huxley and Klein-Gordon equation by the $(G'/G,1/G)$-expansion method. When the parameters take certain special values, different type of waves is derived from the traveling waves. The established results show that the $(G'/G,1/G)$-expansion method is powerful, unified and can be used for many other nonlinear partial differential equations.

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