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FURUTA INEQUALITY FOR CLASS OF 2N-PERINORMAL OPERATORS

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ABSTRACT

In this paper, we introduce the class of (M,2n) and study some applications of Furuta inequality to this class. A bounded linear operator T is said to belong to class (M,2n) if $T^{*2n}T^{2n} \geq (T^{*}T)^{2n}$ for all positive integers n.

Keywords: Furuta inequality, Almost Class (Q), n-perinormal and 2n-perinormal operators.

1. INTRODUCTION

H denotes the seperable complex Hilbert space in this paper, while B(H) is the usual Banach algebra of all bounded linear operators on H. An operator $T \in B(H)$ is said to be perinormal or (M, n) if, $T^{*n}T^{n} \geq (T^{*}T)^{n}$ for all positive integers n. Some results touching on class (M, n) were covered by Hongliang and Fei Zuo in (2) and by Wanjala Victor and Beatrice Obiero in (6). $T \in B(H)$ is said to be 2n- perinormal or (M,2n) if $T^{*2n}T^{2n} \geq (T^{*}T)^{2n}$ for all positive integers n. $T \in B(H)$ is said to be hyponormal if $T^{*}T \geq TT^{*}$ (1), p-hyponormal if $(T^{*}T)^{p} \geq (TT^{*})^{p}$ for $p > 0$ (1). $T \in B(H)$ is said to be in class A (p, p) if $(|T^{*}|^{p}|T|^{2p}|T^{*}|^{p})^{1/2} \geq |T^{*}|^{2p}$ for $p > 0$ and A (p, q) if $(|T^{*}|^{q}|T|^{2p}|T^{*}|^{q})^{q/p+q} \geq |T^{*}|^{2q}$ for $p, q > 0$ (5). $T = U|T|$ is the polar decomposition of T such that $|T| = (T^{*}T)$. The study of Furuta inequality has been widely used in applications in many classes of operators such as but not limited to Class A (p, p), A (p, q) and p-hyponormal operators. Furuta inequalities are stated below (1);

If $A \geq B \geq 0$, then for each $r_o \geq 0$

$(B^{r_o} A^{p_o} B^{r_o})^{1/q_o} \geq (B^{r_o} B^{p_o} B^{r_o})^{1/q_o}$ (1)

$(A^{r_o} A^{p_o} A^{r_o})^{1/q_o} \geq (A^{r_o} B^{p_o} A^{r_o})^{1/q_o}$ (2)

holds for $p_o \geq 0$ and $q_o \geq 1$ and $(1 + 2r_o)q_o \geq p_o + q_o$

2. MAIN RESULTS

Definition 1. Let $T \in B(H)$ be a 2n-perinormal operator if $T^{*2n}T^{2n} \geq (T^{*}T)^{2n}$ for some positive integer n, we denote this class by (M,2n).

The following result follows directly from the definition of 2n-perinormal operator.

Theorem 2. Let ∈ B(H) be a 2n-perinormal operator such that it doubly commutes with an isometric operator say S, then TS is a 2n-perinormal operator.

Proof.

T being a 2n-perinormal implies; $T^{*2n}T^{2n} \geq (T^{*}T)^{2n}$.

Let T doubly commute with S with S being isometric we have;

$ST = TS$, $TS^* = S^*T$ with $S^*S = I$. Setting $P = ST$ we obtain;
$P^{*2n}P^{2n} \geq (P^*P)^{2n}$. Implying TS is a 2n-perinormal.

Theorem 3. Let $T \in B(H)$ be a 2n-perinormal operator, then the following hold;
(i). If T is invertible, T^{-1} is also invertible.
(ii). Any operator $S \in B(H)$ that is unitarily equivalent to T is also 2n-perinormal.
(iii). Suppose $T - k$ is a 2n-perinormal operator for every $k > 0$, then T is a 2nperinormal operator.

Proof.
Proof for (i) and (ii) are straight forward, for (iii), we have;

$$0 \leq (T-k)^{*2n}(T-k)^{2n} - ((T-k)^*(T-k))^{2n} = T^{*2n}T^{2n} - (T^*T)^{2n} - 2k(T^{*2n}T^{n} + T^{*n}T^{2n} - (T^{*2n}T)^2 - (T^*T^{2n})^2) + 4k^2(T^{*n}T^{n} - (T^*T)^{n}).$$

we have;

$$0 \leq \frac{1}{4k2}\{(T-k)^{*2n}(T-k)^{2n} - ((T-k)^*(T-k))^{2n}\} = \frac{1}{4k2}(T^{*2n}T^{2n} - (T^*T)^{2n}) - \frac{1}{2k}(T^{*2n}T^{n} + T^{*n}T^{2n} - (T^{*2}T)^{n} - (T^*T^2)^{n}) + (T^{*n}T^{n} - (T^*T)^{n}).$$

when $k \longrightarrow \infty$ we obtain $T^{*2n}T^{2n} - (T^*T)^{2n} \geq 0$.

Proposition 5. Let K_n be a set of all (M,2n) operators. Then $K_n \subset K_p$ for $0 < p < n$.

Proof. Let $T \in (M,2n)$, then we have;
$T^{*2n}T^{2n} \geq (T^*T)^{2n}$
we have $0 < p/n$. Taking p/n we get;
$(T^*T)^{2p} = (T^{*2n}T^{2n})^{p/n} \leq ((T^*T)^{2p})^{p/n} = (T^*T)^{2p}$.

Theorem 6. If $T \in (M,2n)$ for some positive integer n, then;

- $(|T|^{2n}|T^2|^{2n}|T|^{2n}) \geq |T|^{4n}$.
- $(|T|^{2q}|T^2|^{2n}|T|^{2q})^{q/n+q} \geq |T|^{4q}$ for every $n, q > 0$.
- $|T^2|^{2n} \geq (|T^2|^{n}|T|^{4n}|T^2|^{n})^{1/2}$.
- $|T^2|^{2q} \geq (|T^2|^{q}|T|^{4n}|T^2|^{q})^{q/n+q}$ for every $n, q > 0$.

Proof.
Since $T \in (M,2n)$ and $T = U|T|$ is the polar decomposition of T with U being unitary, we have; $T^{*2n}T^{2n} \geq (T^*T)^{2n}$, implies;
$|T^{2n}| \geq |(T)^{2n}|$
with $|(T)^{2n}| = ((T^*T)^{2n})^{1/2}$
Replacing in Furuta inequality (1) above, we have; Taking $A = |T^{2n}|$, $B = |(T)^{2n}|$,
$r_o = n$, $p_o = 2n$ and $q_o = 2$.
$A \geq B \geq 0$ implies $r_o \geq 0$, $p_o \geq 0$, $q_o \geq 1$.
we equivalently have; $(1 + 2r_o)\, q_o = (1 + 2n)2 = 2 + 4n$. Also;
$p_o + 2r_o = 2n + 2n = 4n$. Thus;
$(1 + 2r_o)\, q_o = p_o + 2r_o$. We then have;
$(|T|^{2n}|T^2|^{2n}|T|^{2n})^{1/2} \geq (|T|^{2n}|T|^{2n}|T|^{2n})^{1/2} = |T|^{2n}$
that is;
$(|T|^{2n}|T^2|^{2n}|T|^{2n})^{1/2} \geq |T|^{2n}$.
Taking the following in Furuta inequality (1);
$A = |T^2|$ and $B = |T|$, $r_o = n$, $q_o = \dfrac{n+q}{q}$, $p_o = 2n$
We have;
$(1 + 2r_o)\, qo - (p_o + 2r_o) = (n + q)\, \{\dfrac{1+2q}{q} - 2\} = \dfrac{n+q}{q} \geq 0$ implies;
$(1 + 2_{ro})q_o \geq p_o + 2r_o$
Furuta inequality implies (1)
$(|T|^{2q}|T^2|^{2n}|T|^{2q})^{q/n+q} \geq (|T|^{2q}|T|^{2n}|T|^{2q})^{q/n+q}$
that is ;
$(|T|^{2q}|T^2|^{2n}|T|^{2q})^{q/n+q} \geq |T|^{2q}$.
Taking the following in Furuta inequality (2)

$A = |T^2|$ and $B = |T|$, $r_o = n$, $q_o = 2, p_o = 2n$;
we obtain ;
$(1 + 2r_o)q_o = 2 + 4n$, $p_o + 2r_o = 4n$, we have $(1 + 2r_o)q_o = p_o + 2r_o$
hence ;
$(|T^2|^n|T^2|^{2n}|T^2|^n)^{1/2} \geq (|T^2|^n|T|^{4n}|T^2|^n)^{1/2}$
That is ;
$|T^2|^n \geq (|T^2|^n|T|^{4n}|T^2|^n)^{1/2}$
Further taking substitutions below in Furuta inequality (2) ;
$A = |T^2|$ and $B = |T|$, $r_o = q$, $q_o = \frac{n+q}{q}$, $p_o = 2n$.
We obtain $(1 + 2r_o)q_o \geq (p_o + 2r_o)$. Thus;
$(|T^2|^q|T^2|^{2n}|T^2|^q)^{q/n+q} \geq (|T^2|^q|T|^{4n}|T^2|^q)^{q/n+q}$
that is;
$|T^2|^{2q} \geq (|T^2|^q|T|^{4n}|T^2|^q)^{q/n+q}$.

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