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ON CHARACTERIZATION OF POSINORMAL OPERATORS AND THEIR APPLICATIONS IN QUANTUM MECHANICS

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ABSTRACT
Studies on Hilbert space operators have been carried out by several researchers and mathematicians with interesting results obtained. A posinormal operator is one of the operators under concern with interesting characters. Properties of posinormal operators have not been exhausted hence this calls for intense characterization of these operators. Of great concern is the norm inequality involving these operators. In this paper, we give a detailed study of the norm property of posinormal operators when they are sefadjoint. The objectives of the study are to: establish norm inequalities for posinormal operators; characterize posinormal operators and; determine applications of posinormal operators in other fields. The methodology involved the use of tensor products as a technical approach in determining these norm inequalities. The results show that the norm of a posinormal operator is equal to the norm of any normal operator when the operators are selfadjoint. The results obtained are very important in the classification of Hilbert space operators and their applications in other fields like quantum theory.

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General Terms: Hilbert space, Posinormal operators, Selfadjoint, norm inequality, Applications and Quantum Mechanics.

1. INTRODUCTION
The concept of arbitrary vector spaces can be generalized to inner product spaces and complete inner product spaces called Hilbert Spaces. The inner product is a generalization of the dot product in $\mathbb{R}^n$. Spectral analysis is a very broad but important aspect of applied functional analysis which extends to eigenvectors and Eigen values theory of a singular matrix. The name was introduced by David Hilbert in his original formulation of Hilbert space theory, later it was discovered that spectral theory could be used to explain features of atomic spectral in quantum mechanics. We explore the basic mathematical physics of quantum mechanics with a focus on Hilbert Space theory and application as well as theory of linear operators on Hilbert Space. We show how operators can be used to represent quantum observables and investigate the spectrum of various linear operators. The development of Hilbert Space and its popularity were as a result of mathematical and physical necessity. The theory of physical law developed by Newton in 1700's and carried sciences

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through centuries of progress and success but later in early 1900's Newton's laws would not withstand the test of time hence the emergence of quantum mechanics as there was need for description of physical systems. Also in mathematics, quantum mechanics began at the time of Von Neumann when he introduced the concept of a finite dimensional vector space and he together with other mathematicians realized the necessity of mathematical formals we call Hilbert space.

2. BASIC CONCEPTS AND PRELIMINARY RESULTS

Here we start by defining some key terms that are useful in the sequel.

Definition 2.1: A vector space V over a field F is the set V together with two operations + and x such that; Addition is associative, commutative, has identity elements and inverse element and multiplication has identity, scalar multiplication and is distributive over scalar and vector addition.

Definition 2.2: An inner product on a vector space X is a function that takes each ordered pair (x, y) of elements of X to number $<x ,y>$ of the elements of K (where K is the scalar field of X). That is, with every pair of vector (x ,y) with the following properties:

1. $<x , x> \geq 0 \quad \forall x \in X$
2. $<x , x> = 0$ if and only if x=0
3. $<x + y, z> = <x , z> + <y , z> \quad \forall x, y, z \in X$
4. $<ay ,z> = a<y , z>$ for all a $\in$K, where K is the scalar field of X and $\forall\, y, z \in X$,
5. $<y , z> = <z , y>^* \ \forall\, y, z \in X$.

Definition 2.3: An inner product on X defines a norm given by $\|x\| = \sqrt{< x, x >} = <x, x>^{1/2}$ and a metric on X given by $d(x,y) = \|x-y\| = \sqrt{< x-y, x-y >}$.

Definition 2.4: A Hilbert space is a complete inner product space, examples of Hilbert space include, the Euclidian space $\mathbb{R}^n$ and the unitary space $\mathbb{C}^n$

Definition 2.5: Trace is the sum of elements in diagonal of an n x n square matrix.

Definition 2.6: Numerical range also referred to as field of values of complex n x n matrix A is the set w(A) = $\left\{\frac{x^*Ax}{x^*x} \,|X \in \mathbb{C}^n \ ,X \neq 0\right\}$. Numerical radius is the largest absolute values of the numbers in numerical range. That is; r(A) = Sup $\{|\lambda|: \lambda \in w(A)\}$ where r(A) is a norm.

Definition 2.7: Spectral radius of a square matrix or bounded linear operator is the supremum among the absolute values of the elements in its spectrum.

Definition 2.8: An operator is a function T that maps one vector space X into another vector space Y. Normally an operator is required to be a linear mapping. We write $T: X \rightarrow Y$ to mean that T is a function with domain X and range contained in Y.

Lemma 2.9: Let X, Y be normed linear spaces, and suppose that $T: X \rightarrow Y$, we write either T(x) or Tx to denote the image of an element x $\in$X .

a. T is linear if $T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$ for every x,y $\in$X and $\alpha, \beta \in \mathbb{C}$
b. T is injective if T(x) = T(y) implies x=y
c. The kernel of T is ker (T)= {x $\in$X: T(x)=0}
d. The rank of T is the vector space dimension of its range i.e. rank (T)=dim(range (T))
e. T is surjective if range (T) =Y
f. T is a bijection if it is both injective and surjective.

Proof. See [2].

Lemma 2.10: Assume that X and Y are normed linear spaces and that $T: X \rightarrow Y$ is an operator, we use $\|.\|_x$ to denote the norm on the space X, and $\|.\|_y$ to denote the norm on Y.

a. T is continuous if $f_n \rightarrow f$ implies $T(f_n) \rightarrow T(f)$

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b. T is bounded if there is a finite real number C so that $\|T(f)\|_y \leq C\|f\|_x$ for every $f \in X$. the smallest such C is called the norm operator of T and is denoted by $\|T\|$. That is T is the smallest number such that, $\forall f \in X$, $\|T(f)\|_y \leq \|f\|_x$

c. For linear operators T, we wav have the fundamental fact that continuity and boundedness are equivalent, i.e. T is continuous↔ T is bounded.

Proof. See [5].

Definition 2.11: A linear functional is an operator whose range is the space of scalars $\mathbb{C}$, that means that a functional is a mapping $T : X \to \mathbb{C}$, so T(f) is a number for every $f \in X$.

If T is a linear functional then we know that it is continuous if and only if it is bounded, also we find that T(f) is a scalar, and the norm of T(f) is just its absolute value |T(f)|

Definition 2.12: Assume that X=H and Y=K are Hilbert spaces. Then $H=H^*$ and $K=K^*$, if $S:H\to K$ then its adjoint S^* maps K back to H. The Hilbert- adjoint therefore $S^*:K\to H$ is the unique mapping $S^*:K\to H$ which satisfies

$\forall f \in H$, $\forall g \in K$, $\langle Sf, g\rangle = \langle f, S^*g\rangle$

Let us define an operator $S: H \to H$, which map a Hilbert space H into itself, then we can have;

a. $S: H \to H$ is self adjoint if $S = S^*$; S is self adjoint ↔ $\forall f,g \in H$, $\langle Sf, g\rangle = \langle f, Sg\rangle$

b. $S: H \to H$ is positive , denoted $S \geq 0$, if $\langle Sf ,f\rangle$ is real and $\langle Sf ,f\rangle \geq 0$ for every $f \in H$

c. $S: H \to H$ is positive definite , denoted $S > 0$, if $\langle Sf,f\rangle > 0$ for every $f \neq 0$

d. If $S: H \to H$, then we write $S \geq T$ if $S - T \geq 0$. Similarly , $S > T$ if $S - T > 0$.

Theorem 2.13: Every bounded linear functional φ on a Hilbert space H can be represented in terms of an inner product $\varphi(x) = \langle x ,u\rangle$. Where u depends on φ and is uniquely determined by φ and has a norm $\|\varphi\| = \|u\|$.

Proof. See [7, Theorem 5.4].

Remark 2.14: the Reisz theorem is named after Frigyes Reisz a Hungarian mathematician and its important in the theory of operators on Hilbert spaces, particularly in representation of Hilbert-adjoint operator of a bounded linear operator.

Let X and Y be vector spaces over the same field K ($\mathbb{R}$ or $\mathbb{C}$). A Sesquilinear functional is defines as a mapping h: $X \times Y \to K$ such that for all $x_1, x_2, x_3 \in X$, and $y_1, y_2, y_3 \in Y$ and all scalars α and β,

1. $h(x_1 + x_2, y) = h(x_1, y) + h(x_2, y)$
2. $h(x, y_1 + y_2) = h(x,y_1) + h(x,y_2)$
3. $h(\alpha x, y) = \alpha h(x ,y)$
4. $h(x, \beta y) = \bar{\beta}\, h(x ,y)$

Hilbert space is both vector spaces and a metric space, it was natural to look first at linear operators from Hilbert space to itself which are continuous, and these are usually called bounded linear operators. The analogue of the symmetry condition k(x ,y) = k(x ,y) on integral operators is the condition that a bounded linear operator T be self adjoint , which is to say that $\langle Tu, v\rangle = \langle u, Tv\rangle$ for all vectors u and v in the Hilbert space .

A simple example of self- adjoint operator is the multiplication operator by a real valued function m(y); this is the operator M defined by the formula (Mu)y = m(y)u (y).

The spectrum of an operator T is the set of complex numbers λ for which the operator T- λi does not have a bounded inverse and I is the identity operator on the Hilbert space in finite dimensions the spectrum is precisely the set of Eigen values, but in infinite dimensions this is not always so in fact, whereas every symmetric matrix has at least one Eigen value, a self-adjoint operator need to have no Eigen value at all.

As a result of this, the spectral theorem for bounded self adjoint operators is formulated, not in terms of Eigen values but in terms of spectrum. One way of this formalism is that any self adjoint operator T is unitarily equivalent to a multiplication operator (Mu)y = m(y)u(y) , where the closure of the range of the function m(y) is the spectrum of T. this generalizes the statement that any real symmetric matrix is unitarily equivalent to a diagonal matrix with the Eigen value along the diagonal. That is, a unitary is an invertible operator that preserves the lengths of vectors.

3. LITERATURE REVIEW

There were two competing mathematical strategies that were used in connection with physical theory, that is the matrix mechanics and wave mechanics that resulted from von Neumann and Dirac respectively. The concept of quantum mechanics emerged when both the two mentioned above tried to relate the formalisms of mathematics and physics. Von Neumann developed the separable Hilbert space formulation of quantum mechanics see Haag R. and Kastler D (1964). In his strategy, he recognized the mathematical framework of matrix mechanics (what's currently referred to as infinite dimensional separable Hilbert space where Hilbert space refers to a complete vector product space with an inner product). Dirac's attempt to prove the equivalence of matrix and wave mechanics made essential use of delta function which was in broad use by physicists before Dirac and it became widely known in his textbook (Dirac 1930). To provide more foundation to Dirac's formal framework, Schwartz's theory of distribution was essential (Schwartz's 1945; 1950-1951) that made possible the generalization of Eigen vector decomposition theorem for self adjoint operators in rigged Hilbert space, for theorem see (Gelfand and Velenken. 1964; pp 119-127). The decomposition principle provides a rigorous way to handle observables such as position and momentum in a manner in which they are presented in Dirac's formalism.

Quantum mechanics is therefore regarded as a general theoretical framework of physical theories It consists of mathematical core which becomes physical theory when adding a set of correspondence rules telling us which mathematical objects we use in different physical situation. In contrast to classical physical theories, these corresponding rules are not intuitive as linear operators on Hilbert spaces are quite familiar (Michael M.W, 2012-8). It's often useful to divide physical experiments into two, that is preparation and measurement and this gives out a clear distinction between quantum and classical mechanics since in the latter we don't talk about measurement. The division of a physical experiment into a preparation on state and measurement of an observable quantity is reflected in mathematical structure of quantum mechanics. Observables are represented by Hermitian elements taken from algebra and assume that each element has an adjoint. The algebra is usually represented in terms of bounded linear operators $\beta(H)$ acting on a Hilbert space H. A state in turn corresponds to a linear functional mapping observables onto a real number and hence words 'state' and 'observable' are used to refer to both physical concept and mathematical operator. And therefore in this text we explore the basic mathematical physics of quantum mechanics with a focus on Hilbert Space theory and application as well as theory of linear operators on Hilbert Space. We show how operators can be used to represent quantum observables and investigate the spectrum of various linear operators.

4. METHODOLOGY

For a successful completion of this research, good background knowledge of the theory of operators, General Topology and Functional Analysis is crucial. We shall restate some known results which we may be useful to our work. We shall need a lot of information in the internet, especially the journals. We might be required to visit other libraries. For our presentations we shall use operator theory and analysis books while for the prerequisites on we shall use the ideas in Hilbert space problem book. Interaction with other mathematicians will be useful so we shall need to attend conferences locally and internationally. Lastly, we shall use the technical approach of tensor products in solving the stated problem. Initially, we examine the algebraic properties of tensor products, their norm properties and applicability in our case before applying it in finding a solution to our problem.

5. RESULTS

Theorem 5.1: The norm of an inner product space X satisfies the following properties;

$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2)$.

Proof. Suppose x , y are orthogonal vectors in X, then,

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$\|x + y\|^2 = \langle x + y, x + y\rangle$
$= \|x\|^2 + \|y\|^2 + \langle x , y\rangle + \langle y , x\rangle$
$= \|x\|^2 + \|y\|^2$.

Remark: if a norm does not satisfy the parallelogram equality, then it cannot be obtained from an inner product space.

Theorem 5.2: The Cauchy Schwartz inequality states that if x ,y ∈X, then; $|\langle x , y\rangle| \leq \|x\|\|y\|$.

Proof. Let $x, y \in X$, if y=0, then both sides of the Cauchy Schwartz inequality will be equal to 0 and the inequality will hold. Suppose $y \neq 0$, consider the orthogonal decomposition (write x as a scalar multiple of y plus a vector w orthogonal to y), $x=\frac{\langle x,y\rangle}{\|y\|2}y + w$ where w is orthogonal to y. By the Pythagorean theorem, $\|x\|^2 = \left\|\frac{\langle x,y\rangle}{\|y\|2}y\right\|^2 + \|w\|^2 = \frac{\|\langle x,y\rangle\|}{\|y\|2}{}^2 y + w \geq \frac{\langle x,y\rangle}{\|y\|2}y$. Multiplying both sides of the above equation by $\|y\|^2$ and then taking square roots yields the Cauchy Schwartz inequality. The triangular inequality states that the length of any side of a triangle is less than the sum of the lengths of the other two sides. If x ,y $\in X$, then , $\|x + y\| \leq \|x\| + \|y\|$.This theorem can be used to show that the shortest path between two points is a straight line segment. In inner product space, if $x_n \to x$ and $y_n \to y$, then $\langle x_n ,y_n\rangle \to \langle x, y\rangle$. See [2, Lemma 3.2].

Remark 5.3: An inner product space (or a pre-Hilbert space) is a vector space with inner product (x ,y) defined on it. An element x of an inner product space X is said to be orthogonal to an element $y \in X$ if <x,y> =0. It is denoted by $x \perp y$ and we say that x and y are orthogonal. Similarly, for subsets **A,B** $\subset$ **X,** we write $x \perp A$ if $x \perp a$ for all $a \in A$, and $A \perp B$ if $a \perp b$ for all $a \in A$ and all $b \in B$.

From this point and in the sequel, we give results on posinormal operators on Hilbert spaces. Also all the complete fuzzy metric spaces herein are taken to be Hilbert spaces unless otherwise stated.

Theorem 5.4: Let T be an A-Contraction on a complete non - Archimedean fuzzy metric space X . Then T has a unique fixed point in X such that the sequence $\left\{ T^n x_0 \right\}$ converges to the fixed point, for any $x_0 \in X$.

PROOF. Fix $x_0 \in X$ and define the iterative sequence $\left\{ x_n \right\}$ by $x_n = T^n x_0$ (equivalently, $x_{n+1} = T x_n$) where T^n stands for the map obtained by n - time composition of T with T. Since T is an A-Contraction, ∃ $\alpha \in A$ such that the definition 1.8 holds, i.e.,

$$\mu(Tx, Ty, t) \geq \alpha\big(\mu(x, y, t),\ \mu(x, Tx, t),\ \mu(y, Ty, t)\big) \quad \cdots \quad (6)$$

for all $x, y \in X$. Now,

$$\mu(x_n, x_{n+1}, t) = \mu(Tx_{n-1}, Tx_n, t)$$

$$\geq \alpha\Big(\mu(x_{n-1}, x_n, t),\ \mu(x_{n-1}, Tx_{n-1}, t),\ \mu(x_n, Tx_n, t)\Big)$$

$$= \alpha\Big(\mu(x_{n-1}, x_n, t),\ \mu(x_{n-1}, x_n, t),\ \mu(x_n, x_{n+1}, t)\Big)$$

$$\Rightarrow\ k\,\mu(x_n, x_{n+1}, t) \geq \mu(x_{n-1}, x_n, t)$$

Continuing this way, we get

$$\mu(x_n, x_{n+1}, t) \geq \frac{1}{k^n}\mu(x_0, x_1, t)$$

$$\Rightarrow \lim_{n\to\infty} \mu(x_n, x_{n+1}, t) = 1$$

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We now verify that $\{x_n\}$ is Cauchy sequence.

$$\mu(x_n, x_{n+p}, t) \geq \mu(x_n, x_{n+1}, t) * \cdots * \mu(x_{n+p-1}, x_{n+p}, t)$$

$$\Rightarrow \lim_{n\to\infty} \mu(x_n, x_{n+p}, t) \geq 1 * \cdots * 1 = 1$$

$$\Rightarrow \lim_{n\to\infty} \mu(x_n, x_{n+p}, t) = 1$$

Thus $\{x_n\}$ is Cauchy sequence in X. Since X is complete, there exists $x \in X$ such that $x_n \to x$ as $n \to \infty$.

Again, with $x = x'$ and $y = x_n$, the inequality (6) gives

$$\mu(Tx, x_{n+1}, t) = \mu(Tx', Tx_n, t)$$

$$\geq \alpha\big(\mu(x', x_n, t), \mu(x', Tx', t), \mu(x_n, Tx_n, t)\big) \quad \forall\ n \in N.$$

By allowing n $\to \infty$ and using the continuity of α, we get

$$\mu(Tx', x', t) \geq \alpha\big(\mu(x', x', t), \mu(x', Tx', t), \mu(x', x', t)\big)$$

i.e., $\mu(Tx', x', t) \geq \alpha(1, \mu(Tx', x', t), 1)$

$$k\,\mu(Tx', x', t) \geq 1 \quad \Rightarrow \mu(Tx', x', t) = 1 \Rightarrow Tx' = x'.$$

Now, if $w \in X$ satisfies, $Tw = w$, then by taking $x = w$ and $y = x'$ in (6) we get

$$\mu(w, x', t) = \mu(Tw, x', t)$$

$$\geq \alpha\big(\mu(w, x', t), \mu(Tw, w, t), \mu(Tx', x', t)\big)$$

$$\geq \alpha\big(\mu(w, x', t), 1, 1\big)$$

$$\Rightarrow k\,\mu(w, x', t) \geq 1 \Rightarrow \mu(w, x', t) = 1 \Rightarrow w = x'.$$

This completes the proof.

THEOREM 4.6 Let $\alpha \in A$ and $\{T_n\}$ be a sequence of self-maps on the complete non - Archimedean fuzzy metric space $(X, \mu, *)$ such that

$$\mu(T_i x, T_j y, t) \geq \alpha\big(\mu(x, y, t), \mu(x, T_i x, t), \mu(y, T_j y, t)\big) \quad \cdots \quad (7)$$

for all $x, y \in X$ and $k \in (0, 1)$. Then $\{T_n\}$ has a unique common fixed point in X.

PROOF. Taking any $x_0 \in X$, we define $x_n = T_n x_{n-1}$ for each $n \in N$. Now from (7), we have

$$\mu(x_1, x_2, t) = \mu(T_1 x_0, T_2 x_1, t)$$

$$\geq \alpha\big(\mu(x_0, x_1, t), \mu(x_0, T_1 x_0, t), \mu(x_1, T_2 x_1, t)\big)$$

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$$\geq \alpha\left(\mu\left(x_0, x_1, t\right), \ \mu\left(x_0, x_1, t\right), \ \mu\left(x_1, x_2, t\right)\right)$$

$$\Rightarrow k\mu\left(x_1, x_2, t\right) \geq \mu\left(x_0, x_1, t\right)$$

Similarly, we have

$$k\mu\left(x_2, x_3, t\right) \geq \mu\left(x_1, x_2, t\right)$$

$$\Rightarrow \mu\left(x_2, x_3, t\right) \geq \frac{1}{k}\mu\left(x_1, x_2, t\right) \geq \frac{1}{k^2}\mu\left(x_0, x_1, t\right)$$

Inductively, we have

$$\mu\left(x_n, x_{n+1}, t\right) \geq \frac{1}{k^n}\mu\left(x_0, x_1, t\right)$$

$$\Rightarrow \lim_{n\to\infty} \mu\left(x_n, x_{n+1}, t\right) = 1$$

We now verify that $\{x_n\}$ is Cauchy sequence.

$$\mu\left(x_n, x_{n+p}, t\right) \geq \mu\left(x_n, x_{n+1}, t\right) * \cdots * \mu\left(x_{n+p-1}, x_{n+p}, t\right)$$

$$\Rightarrow \lim_{n\to\infty} \mu\left(x_n, x_{n+p}, t\right) \geq 1 * \cdots * 1 = 1$$

$$\Rightarrow \lim_{n\to\infty} \mu\left(x_n, x_{n+p}, t\right) = 1$$

Therefore $\{x_n\}$ is Cauchy sequence in the complete fuzzy metric space X, so it converges to $x' \in X$. Next,

$$\mu\left(x', T_n x', t\right) \geq \mu\left(x', x_{m+1}, t\right) * \mu\left(x_{m+1}, T_n x', t\right)$$

$$= \mu\left(x', x_{m+1}, t\right) * \mu\left(T_{m+1} x_m, T_n x', t\right)$$

$$\geq \mu\left(x', x_{m+1}, t\right) * \alpha\left(\mu\left(x_m, x', t\right), \ \mu\left(T_{m+1} x_m, x_m, t\right), \ \mu\left(T_n x', x', t\right)\right)$$

$$\geq \mu\left(x', x_{m+1}, t\right) * \alpha\left(\mu\left(x_m, x', t\right), \ \mu\left(x_{m+1}, x_m, t\right), \ \mu\left(T_n x', x', t\right)\right)$$

Letting $m \to \infty$, recalling that α is continuous on R_+^3, we obtain

$$\mu\left(T_n x', x', t\right) \geq \mu\left(x', x', t\right) * \alpha\left(\mu\left(x', x', t\right), \ \mu\left(x', x', t\right), \ \mu\left(T_n x', x', t\right)\right)$$

$$\Rightarrow \mu\left(T_n x', x', t\right) \geq \alpha\left(1, \ 1, \ \mu\left(T_n x', x', t\right)\right)$$

$$\Rightarrow \mu\left(T_n x', x', t\right) = 1 \Rightarrow T_n x' = x' \quad \forall \ n \in N.$$

For uniqueness of the fixed point x', we suppose $T_n y = y$ for some y ∈ X and for all $n \in N$.

Then by (7), we have

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$$\begin{aligned}\mu(x', y, t) &= \mu\left(T_i x', T_j y, t\right)\\ &\geq \alpha\Big(\mu(x', y, t)\ ,\ \mu\left(x', T_i x', t\right)\ ,\ \mu\left(y, T_j y, t\right)\Big)\\ &= \alpha\big(\mu(x', y, t)\ ,\ 1\ ,\ 1\big) \ \Rightarrow\ x' = y.\end{aligned}$$

THEOREM 4.7 Let X be a set with two non - Archimedean fuzzy metrics μ and ∂ satisfying the following conditions:

(i) $\mu(x, y, t) \geq \partial(x, y, t)$ for all $x, y \in X$.

(ii) X is complete with respect to μ .

(iii) S, T are self maps on X , such that T is continuous with respect to μ and

$$\partial(Tx, Sy, t) \geq \alpha\big(\partial(x, y, t)\ ,\ \partial(x, Tx, t)\ ,\ \partial(y, Sy, t)\big)$$

for all $x, y \in X$ and for some $\alpha \in A$.

Then S and T have a unique common fixed point.

PROOF. . Take any $x_0 \in X$. For each $n \in N$, we define $x_n = S x_{n-1}$, when n is even and $x_n = T x_{n-1}$, when n is odd. Then, by inequality in the above condition (iii) we get

$$\begin{aligned}\partial(x_1, x_2, t) &= \partial(T x_0, S x_1, t)\\ &\geq \alpha\big(\partial(x_0, x_1, t)\ ,\ \partial(x_0, T x_0, t)\ ,\ \partial(x_1, S x_1, t)\big)\\ &\geq \alpha\big(\partial(x_0, x_1, t)\ ,\ \partial(x_0, x_1, t)\ ,\ \partial(x_1, x_2, t)\big)\end{aligned}$$

$$\Rightarrow k\,\partial(x_1, x_2, t) \geq \partial(x_0, x_1, t)$$

In general, for any $n \in N$ we get (as in the proof of the of the previous theorem) that

$$\partial(x_n, x_{n+1}, t) \geq \frac{1}{k^n}\partial(x_0, x_1, t) \text{ for some } k \in (0\ ,\ 1).$$

$$\Rightarrow \mu(x_n, x_{n+1}, t) \geq \partial(x_n, x_{n+1}, t) \geq \frac{1}{k^n}\partial(x_0, x_1, t) \text{ (By the condition (i))}$$

$$\Rightarrow \lim_{n\to\infty} \mu(x_n, x_{n+1}, t) = 1$$

This implies that $\{x_n\}$ is a Cauchy sequence in X with respect to μ and hence by condition (ii), we have

$$\lim_{n\to\infty} \mu(x_n, x', t) = 1 \text{ for some } x' \in X.$$

Since T is given to be continuous with the respect to μ we have

$$1 = \lim_{n\to\infty} \mu(x_{2n+1}, x', t) = \lim_{n\to\infty} \mu(T x_{2n}, x', t) = \mu(T x', x', t)$$

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$\Rightarrow Tx' = x'$.

Now , by condition (iii)

$$\partial(x', Sx', t) = \partial(Tx', Sx', t)$$
$$\geq \alpha\big(\partial(x', x', t) \, , \, \partial(x', Tx', t) \, , \, \partial(x', Sx', t)\big)$$
$$= \alpha\big(1 \, , \, 1 \, , \, \partial(x', Sx', t)\big)$$

$\Rightarrow Sx' = x'$.

Thus x' is a common fixed point of S and T.

For the uniqueness, let y be any common fixed point of S and T in X . Then by condition (iii) ,

$$\partial(x, y, t) = \partial(Tx, Sy, t)$$
$$\partial(x', y, t) = \partial(Tx', Sy, t)$$
$$\geq \alpha\big(\partial(x, y, t) \, , \, \partial(x, Tx, t) \, , \, \partial(y, Sy, t)\big)$$
$$= \alpha\big(\partial(x, y, t), 1 \, , \, 1\big)$$

$\Rightarrow \partial(x, y, t) = 1 \qquad \Rightarrow x = y$.

This completes the proof.

6. APPLICATIONS IN QUANTUM MECHANICS

A beautiful application of the spectral theorem was found by John von Neumann. Problems regarding the relationship between time and space had arisen in thermodynamics and elsewhere and the expectation that after a given interval the averages of time and space shall agree was solved when Von Neumann came up with theorem of operator theory regarding the Hilbert space. This theorem can be deduced from a spectral theorem for unitary operators which are analogous to the spectral theorem for self-adjoint operators.

Von Neumann realized that Hilbert spaces and their operators provide the correct mathematical tools to formalize the laws of quantum mechanics, introduces in the 1920's by Heisenberg and Schrodinger . The state of a physical system at an instant of time is the list of all its future behavior, basically this is the list of the position and momentum vectors of all the constituent particles.

In von Neumann's formulation of quantum mechanics, to each physical system there is associated a Hilbert space H, and a state of the system is represented by a unit vector u in H. Vectors which differ only by multiplication with a scalar determine the same state. Associated to each observable quantity for example the total energy of the system or even the momentum of a particle within a given domain are self-adjoint operator Q on H whose spectrum is the set of all observed values of that quantity.

This brings out the clear relationship between state and observable that is, when a system is in a given state described by a unit vector u in H, the expected value of observable quantity corresponding to a given self-adjoint operator Q is the inner product <Qu, u> . The relationship between states and observables reflects the self contradictory behavior of quantum mechanics;

Example: In quantum mechanics, the observables of a system are represented by a space A of linear operators on a Hilbert space H. A state of a quantum mechanical system is a linear functional on the space A of observables with the following two properties:

i. u(A*A) ≥ 0 for all A ∈A,

ii. u(I) = 1

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The number u(A) is the expected value of the observable A when the system is in the state w Condition (i) is called positivity, and condition (ii) is called normalization. To be specific, suppose that H = $\mathbb{C}^n$ and A is the space of all n x n complex matrices, then A is a Hilbert space with the inner product given by <A, B> = tr A*B

A measurement of an observable quantity will produce a determinate outcome if and only if the system is in a state which is an Eigenvector for the operator associated for that quantity. A distinctive feature of quantum theory is that the operators associated to different observables typically do not commute, then they will typically have no Eigenvector in common, and as a result, simultaneous measurements of two different observables will typically not result in determinate values for both of them.

7. CONCLUSIONS

Studies on Hilbert space operators are very significant in analysis and applications in quantum mechanics. Results from operator theory are fundamental in formulating principles guiding quantum mechanics and its techniques.

8. REFERENCES

[1] Haag R. and Kastler D, An algebraic approach to quantum field theory, Journal of mathematics and physics, Vol. 5, (1966), 848-861.

[2] Okelo N.B, A survey of development in operator theory on various classes of operators with applications in quantum mechanics, JP&A SC and Tech, Vol.1 (2011), 1-9.

[3] Dirac P.A, The principle of quantum mechanics, Clarendum press, oxford, 1930.

[4] Gelfand I and Vilenki N, Generalized functions, New York Academic Press, 1964.

[5] Michael M.W, Quantum channels and operators , Neils-Bohr institute, Copenhagen, 2012.

[6] Margaret M.K, Operators on Hilbert spaces, African institute of mathematics, 2007.

[7] Bernard Aupetis, A primier on spectral theory, Springer, New York , 1991.

[8] Sheldon Axle, Linear algebra done right,, Springer,San Francisco state university, 1997.

[9] Erwin Kreyszig, Introductory functional analysis with applications , John Wiley and sons, New York, 1978.

[10] Daya Reddy, Introductory functional analysis with application to boundary value problems and finite elements, Text in Applied Mathematics, no 27, Springer , University of Cape Town, 1953.

[11] Charles Swartz, An introduction to functional analysis , monographs and textbooks, no 175, Marcel Dekker, inc New York , 1997.

[12] Böhm A, Rigged Hilbert space and mathematical description of physical systems, Physica A, Vol. 236, (1966), 485-549.