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ON CLASS (BQ) OPERATORS

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ABSTRACT

In this paper, we introduce the class of (BQ) operators acting on a complex Hilbert space H. An operator if $T \in B(H)$ is said to belong to class (BQ) if $T^{*2}T^2$ commutes with $(T^*T)^2$ or $[T^{*2}T^2, (T^*T)^2] = 0$. We investigate some properties that this class is privileged to have. We analyze the relation of this class to class (Q) and then generalize it to class (nBQ) and analyze its relation to the class of n-power class (Q) through complex symmetric operators.

Keywords: Class (Q),Normal ,Binormal operators , complex symmetric operators , n-power class (Q) , (BQ) operators.

1. INTRODUCTION

Throughout this paper , H denotes the usual Hilbert space over the complex field and B(H) the Banach algebra of all bounded linear algebra on an infinite dimensional separable Hilbert space H . An operator $T \in B(H)$ is said to be class (Q) if $T^{*2}T^2 = (T^*T)^2$ (1). Over the recent years, this class has been expounded into almost class (Q) (4), n-power class (Q) (2), quasi M class (Q) and (α, β)-class (Q) (3) among others. An operator $T \in B(H)$ is said to belong to class (BQ) if $T^{*2}T^2(T^*T)^2 = (T^*T)^2T^{*2}T^2$. A conjugation on a Hilbert space H is an anti-linear operator C from Hilbert space H onto itself that satisfies $C\xi, C\zeta i = h\zeta, \xi i$ for every $\xi, \zeta \in H$ and $C^2 = I$. An operator T is said to be complex symmetric if $T = CT^*C$.

2. MAIN RESULTS

Theorem 1. Let $T \in B(H)$ be such that $T \in B(Q)$, then the following are also true for (BQ);
(i). λT for any real λ
(ii). Any $S \in B(H)$ that is unitarily equivalent to T.
(iii). The restriction T-M to any closed subspace M of H.
Proof. (i). The proof is trivial.
(ii). Let $S \in B(H)$ be unitarily equivalent to T, then there exists a unitary operator $U \in B(H)$ with

$S = U^*TU$ and $S^* = U^*T^*U$. Since $T \in B(Q)$, we have;
$T^{*2}T^2(T^*T)^2 = (T^*T)^2T^{*2}T^2$, hence
$S^{*2}S^2(S^*S)^2 = UT^{*2}U^*UT^2U^*(UT^*U^*UTU^*)^2$
$= UT^{*2}U^*U^*T^2U^*UT^*U^*UT^*U^*UTU^*UTU^*$
$= UT^{*2}T^2(T^*T)^2U^*$
$= U(T^*T)^2T^{*2}T^2U^*$
and

$(S^*S)^2 S^{*2} S^2 = (UT^*U^*UTU^*)^2 UT^{*2}U^*UT^2U^*$
$= UT^*U^*UTU^*UT^*U*UTU^*UT^{*2}U^*UT^2U^*$
$= UT^*TT^*TT^{*2}T^{*2}U^*$
$=U(T^*T)^2 T^{*2} T^2 U^*$
Thus S is unitarily equivalent to T.

(iii) . If T is in class (BQ), then;

$T^{*2} T^2 (T^*T)^2 = (T^*T)^2 T^{*2} T^2.$
Hence;
$(T/M)^{*2} (T/M)^2 \{(T/M)^* (T/M)\}^2$
$=(T/M)^2 (T/M)^2\{(T/M)^* (T/M)\}^2$
$= (T^{*2}/M) (T^2/M) \{(T^*/M) (T/M)\} \{(T^*/M) (T/M)\}$
$= \{(T^*T)^2/M\} \{T^{*2}T^2/M\}$
$= \{(T^*/M) (T/M)\}^2 (T/M)^{*2} (T/M)^2$
Thus $T/M \in (BQ)$.

Theorem 2. If $T \in B(H)$ is in Class (Q), then $T \in (BQ)$.

Proof. If $T \in (Q)$, then
$T^{*2}T^2 = (T^*T)^2$
post multiplying both sides by $T^{*2} T^2$;
$T^{*2}T^2 T^{*2} T^2= (T^*T)^2 T^{*2} T^2$
$T^{*2}T^2 T^*TT^*T= (T^*T)^2 T^{*2} T^2$
$T^{*2} T^2 (T^*T)^2 = (T^*T)^2 T^{*2} T^2.$

Theorem 3. Let $S \in (BQ)$ and $T \in (BQ)$. If both S and T are doubly commuting, then ST is in (BQ).

Proof.
$(ST)^{*2} (ST)^2 ((ST)^* (ST))^2$
$=S^{*2} T^{*2} S^2 T^2 ((ST)^* (ST))((ST)^*(ST))$
$=S^{*2} T^{*2} S^2 T^2 ((S^*T^*)(ST))((S^*T^*)(ST))$
$=S^{*2} T^{*2} S^2 T^2 S^* T^* STS^*T^* STS^* T^* ST$
$= S^{*2} T^{*2} S^2 T^2 S^* ST^*TS^* ST^* T$
$=T^{*2} T^2 S^{*2} S^2 S^* SS^* ST^*TT^*T$
$=T^{*2} T^2 S^{*2} S^2 (S^*S)^2T^*TT^*T$
$=T^{*2} T^2(S^*S)^2 S^{*2} S^2 T^* TT^*T$ (Since $S \in (BQ)$).
$= (S^*S)^2 T^{*2} T^2 T^* TT^*TS^{*2} S^2$
$=(S^*S)^2 T^{*2} T^2 (T^*T)^2 S^{*2} S^2$
$=(S^*S)^2 (T^*T)^2 T^{*2} T^2 S^{*2} S^2$ (Since $T \in (BQ)$).
$=((S^*S)(T^*T))^2 T^{*2} S^{*2} T^2 S^2$
$=((S^*T^*)(ST))^2 S^{*2} T^{*2} S^2 T^2$
$= ((ST)^*(ST))^2 (ST)^{*2} (ST)^2$
Thus $ST \in (BQ)$.

Theorem 4. Let $T \in B(H)$ be a class (BQ) operator such that $T = CT^*C$ with C being a conjugation on H. If C is such that it commutes with $T^{*2} T^2$ and $(T^*T)^2$, then T is a class (Q) operator.

Proof. Let $T \in (BQ)$ and complex symmetric, then we have; $T^{*2} T^2 (T^*T)^2 = (T^*T)^2 T^{*2} T^2$ and $T = CT^*C$.
hence;
$T^{*2} T^2(T^*T)^2 = (T^*T)^2 T^{*2} T^2$
$T^{*2}T^2 CTCCT^*CCTCCT^*C = (T^*T)^2 CTCCT^*CCTCCT^*C.$
$T^{*2} T^2 CTT^* TT^* C = (T^*T)^2 CTT^*TT^* C$
$T^{*2} T^2 CT^2 T^{*2} C = (T^*T)^2 CT^*TT^*TC$
$T^{*2}T^2 CT^{*2} T^2 C = (T^*T)^2 C(T^*T)^2 C.$
C commutes with $T^{*2} T^2$ and $(T^*T)^2$ hence we obtain ;
$T^{*2}T^2T^{*2}T^2 = (T^*T)^2 (T^*T)^2.$

which implies ;
$T^{*2}T^{2} = (T^{*}T)^{2}$ and thus $T \in (Q)$.

Definition 5. An operator T is said to be in class (nBQ) if $T^{*2}T^{2n}(T^{*}T^{n})^{2} = (T^{*}T^{n})^{2}T^{*2}T^{2n}$ for a positive integer n.

Theorem 6. Let $T \in B(H)$ be (n-1)-class (Q) operator, if T is a complex symmetric operator such that C commutes with $(T^{*}T)^{2}$, then T is an n-power class (Q) operator.

Proof. With T being complex symmetric and (n-1)-class (Q) , we have ;
$T = CT^{*}C$ and $T^{*2}T^{2n-2} = (T^{*}T^{n-1})^{2}$.
We obtain ;
$T^{*2}T^{2n-2}T^{2} = (T^{*}T^{n-1})^{2}T^{2}$.
hence ;
$T^{*2}T^{2n} = (T^{*}T^{n-1})^{2}T^{2}$.
$T^{*2}T^{2n} = T^{*2}T^{2n-2}T^{2} = T^{2n-2}T^{*2}T^{2}$
$T^{*2}T^{2n} = T^{2n-2}T^{*}T^{*}TT = T^{2n-2}CTCCTCCT^{*}CCT^{*}C = T^{2n-2}CTTT^{*}T^{*}C$.
$=T^{*2}T^{2n} = T^{2n-2}CT^{2}T^{*2}C = T^{2n-2}C(T^{*}T)^{2}C$
Since C commutes with $(T^{*}T)^{2}$ we obtain ;
$T^{*2}T^{2n} = T^{2n-2}(T^{*}T)^{2}CC = T^{2n-2}T^{*2}T^{2}CC = T^{2n-2}T^{2}T^{*2}CC = T^{*2}T^{2n} = (T^{*}T^{n})^{2}$
Hence T is n-power class (Q).

REFERENCES

[1] **Jibril, A.A.S..,** On Operators for which T ∗2(T)2 = (T ∗T)2, international mathematical forum, vol. 5(46) ,2255-2262.

[2] **S. Paramesh, D. Hemalatha and V.J. Nirmala.,** A study On n-power class (Q) operators, international research journal of engineering and technology, vol.6(1), (2019) , 2395-0056.

[3] **wanjala Victor and A.M. Nyongesa., On** (α, β)-class (Q) Operators, international journal of mathematics and its applications, vol. 9(2) (2021), 111-113.

[4] **Wanjala Victor and Beatrice Adhiambo Obiero.,** On almost class (Q) and class (M,n) operators ,international journal of mathematics and its applications ,vol . 9(2) (2021), 115-118.